Kappa Performance Measures with Johnson Distributions

Naceur Naguez[a] and Jean-Luc Prigent[b]

[a] *THEMA, University of Cergy-Pontoise, 33 bd du Port, 95011, Cergy, France*
Naceur.Naguez@u-cergy.fr
[b] *THEMA, University of Cergy-Pontoise, 33 bd du Port, 95011, Cergy, France*
Jean-luc.Prigent@u-cergy.fr

ABSTRACT

In this paper, we analyze the Kappa performance measures of portfolio returns having Johnson distributions. Kappa performance measures are based on downside risk measures, which better allows evaluating risk and performance of complex returns such as those of hedge funds. These measures take account of the whole return distribution. We illustrate how they evolve according to the four basic parameters of the Johnson distributions.

I.   INTRODUCTION

The empirical analysis of hedge fund returns has shown that the mean-variance approach is not adequate to study the risk and performance of these funds. Using monthly data, Fung and Hsieh (1997), Brooks and Kat (2001), Amenc et al. (2002, 2003) demonstrate that distributions of hedge funds returns have a negative skewness and an excess kurtosis. This leads to conclude that, for this type of fund, the challenge lies not only in the first two moments but also in higher moments. In this context, the selection of a family of distributions with sufficient flexibility to take account of the asymmetry and the kurtosis is necessary to study the behaviour of hedge funds returns. The Johnson distribution system (1949) satisfies this requirement. As mentioned in Passow (2004), this family of probability distributions can calibrate independently the four first moments, "while disregarding potentially insignificant moments of order higher than 5." The introduction of Johnson distributions allows covering up to fourth-order moment patterns of (Log-) Normal, (2-sided) Student t and Weibull distributions. The family of Edgeworth expansions and twelve different types of Pearson distributions are also included. Note that these latter distributions include 1st and 2nd kind Beta and Gamma distributions. This is the reason why Johnson distributions can actually model the specific moment patterns.

Since 2000, portfolio management theory has been based in particular on downside risk measures. In line with Basel II accords for banking regulations, these latter ones are related to determination of economical capital allocation (see Goovaerts et al., 2002). As illustrated also by Jarrow and Zhao (2006), there exist significant differences between mean-variance optimal portfolios and those based on lower partial moments, when asset returns are far from being Gaussian.

The literature about financial risk and performance measurement has increased continuously. The first and basic performance measures for asset management are the Sharpe's ratio, the Treynor's ratio and the Jensen's Alpha. However, we must often deal with asymmetric return distributions having also fat tails. Consequently, it is necessary to introduce performance measures that are usually based on "reward/risk" ratios. Additionally, these risk measures must involve the whole return distribution, in particular the probabilities to get significant negative returns. For this purpose, downside risk measures have been introduced (see e.g. Pedersen and Satchell, 1998; Artzner et al., 1999; Szegö, 2002).[1] Then, using the downside lower partial moment, Keating and Shadwick (2002) define a new performance measure, called the Omega measure. This performance measure is based on a gain-loss approach. It takes account of investor loss aversion, as in Tversky and Kahneman (1992). This performance measure is defined as the ratio of the expectation of gains above the threshold and the expectation of losses below the threshold. Farinelli and Tibiletti (2008) and Zakamouline and Koekebakker (2009) introduce generalized Sharpe ratios to evaluate portfolio performance. As proved by Pedersen and Satchell (1998, 2002), the Sortino ratio (which corresponds to Kappa (2)) is linked to utility function with lower risk aversion. Zakamouline (2010) generalizes this result by showing that Kappa measures correspond to performance measures associated to piecewise linear plus power utility functions. Kappa (n) measures have been used to examine performance of a large class of financial models, in particular to deal with hedge fund style or structured equity funds (see Bertand and Prigent, 2011). We examine how these measures behave according to the parameters of Johnson

distributions.

This paper is organized as follows. Section II provides general results about Kappa performance measures. Section III is a survey about Johnson distributions. We also develop an empirical illustration of such probability distributions and illustrate in particular the relationship between their four parameters and their four moments. Section IV shows how the Kappa measures depend on these four parameters. Such results can potentially be applied to examine what hedge fund returns can yield to the best Kappa ranking. Section V provides numerical illustrations. Section VI contents the main conclusions.

II.   THE KAPPA PERFORMANCE MEASURES

A.   Definition and General Properties

As mentioned by Unser (2002), individuals are often only interested in an evaluation of outcomes which have values smaller than a given target.[2] This feature yields to introduce downside risk measures (see, e.g. Ebert, 2005).

The Kappa measures, introduced by Kaplan and Knowles (2004), are defined by: for $l=1,2,...,$

$$Kappa_l(L) = \frac{E_p[X]-L}{\left(E_p\left[\left[(L-X)^+\right]^l\right]\right)^{\frac{1}{l}}}.$$

For $l=1$, we get the Sharpe Omega measure and, for $l=2$, it is the Sortino ratio. Zakamouline (2010) shows that Kappa measures correspond to performance measures based on piecewise linear plus power utility functions. Indeed, consider the following utility function:

$$U_L(v) = (v-L)^+ - \left(\left[(L-v)^+\right]\right) + \frac{\Phi}{n}\left[(L-v)^n\right],$$

with $\Phi>1$ and n a nonzero integer. Then, the investor's capital allocation problem, examined by Zakamouline (2010) yields to the following utility of the optimal allocation:

$$E\left[U_L^*(V)\right] = \frac{n-1}{n}\left(\frac{E[V]-L}{\left(E\left[(L-V)^{+n}\right]\right)^{\frac{1}{n}}}\right)^{\frac{n}{n-1}},$$

Since $E\left[U_L^*(V)\right]$ is an increasing transformation of the Kappa (n) ratio, this latter one is actually based on the utility U_L

Remark 1 Note that U_L is convex on $[-\infty, L]$ if and only if $n=1$. This case corresponds to the Omega measure, obtained as a limiting case when $n\to1$. Thus, the Omega measure is related to the maximization of an expected utility with loss aversion, as introduced by Tversky and Kahneman (1992).

B.   A Special Case: The Omega Ratio

Keating and Shadwick (2002) and Cascon et al. (2003) have introduced the Omega performance measure to take the whole return distribution into account, without requiring any parametric assumption on the return distribution. Omega is defined as the probability weighted ratio of gains to losses relative to a return threshold. It is given by:

$$\Omega_X(L) = \frac{\int_L^b (1 - F(x)) dx}{\int_a^L F(x) dx},$$

where $F(.)$ is the cumulative distribution function of the asset return X defined on the interval (a,b), with respect to the probability distribution P and L is the return threshold selected by the investor. Returns below this threshold are considered as losses and returns above as gains. At a given return threshold, the investor should always prefer the portfolio with the highest Omega value. The Omega function can be written as (see Kazemi et al., 2004):

$$\Omega_x(L) = \frac{E_P\left[(X-L)^+\right]}{E_P\left[(L-X)^+\right]}.$$

It is the ratio of the expectations of gains above the threshold L to the expectations of the losses below the threshold L.[3]   Kazemi et al. (2004) introduce also the Sharpe Omega measure as:

$$Sharpe_\Omega(L) = \frac{E_P[X] - L}{E_P\left[(L-X)^+\right]} = \Omega_X(L) - 1.$$

Note that, if $E_P[X] < L,$ the Sharpe Omega will be negative, otherwise it will be positive. If $L = E_P[X]$, then $\Omega_X(L) = 1$, where $\Omega_X(.)$ is a monotone decreasing function. $\Omega_X(.) = \Omega_Y(.)$ if and only if $F_X = F_Y$. The level of threshold must be specified according to investment objective and individual risk aversion.

III.   THE JOHNSON DISTRIBUTIONS

A.   Definition

The Johnson system covers three main families of distributions that can model a wide variety of empirical distributions. Given a continuous random variable X, X has a Johnson distribution if there exists a monotonic function g such that:

$$Z = \gamma + \delta g\left(\frac{X - \xi}{\lambda}\right); \delta > 0; \lambda > 0.$$

where Z is a standard normal random variable, and $\gamma, \delta, \xi, \lambda$ are given parameters.

Usually γ and δ correspond to trend parameters, ξ is a positional parameter and finally λ corresponds to a scaling parameter. In our case, X denotes financial asset returns,[4] specifically hedge funds.

Notation: We introduce the function h and its inverse by setting:

$$h(x) \;=\; \gamma + \delta g\left(\frac{x-\xi}{\lambda}\right),$$

$$h^{-1}(z) = x = \lambda g^{-1}\left(\frac{z-\gamma}{\delta}\right) + \xi.$$

The function *g (.)* is generally one of the three following functions defined by:

- Lognormal distribution $S_L \;:\; g(y) = \ln(y),$

- Unbounded distribution $S_U \;:\; g(y) = \ln\left(y + \sqrt{(1+y^2)}\right) = \sinh^{-1}(y),$

- Bounded distribution $S_B \;:\; g(y) = \ln\left(\frac{y}{1-y}\right).$

with $y = \left(\dfrac{x-\xi}{\lambda}\right).$

The S_U distribution is unbounded in both directions while the S_B distribution is bounded in both directions *$(\xi, \xi+\lambda)$.*The Johnson S_U distribution is most interesting because its support is the set of real numbers. In addition, it offers, by the number of parameters characterizing a wide class of asymmetries and flattening.

This distribution can model a large number of financial products. Therefore, in what follows, we focus in mainly on this distribution.

The Johnson probability density function (pdf) f_X of S_U distribution is easily deduced from standard normal distribution. It is given by canonical transformations of the standard Gauss distribution:

$$f_X(x) = \frac{\delta}{\lambda\sqrt{2\pi}\sqrt{\left(\frac{x-\xi}{\lambda}\right)^2 + 1}} \exp\left[-\frac{1}{2}\left\{\gamma + \delta\ln\left(\frac{x-\xi}{\lambda} + \sqrt{\left(\frac{x-\xi}{\lambda}\right)^2 + 1}\right)\right\}^2\right]$$

When using Johnson distributions, we can for example search for one of the main Johnson distributions. Therefore, in that case, the first step is to determine which of the three families would be used. The second step is to estimate the parameters of this distribution.

We can also determine the best function g, using a non parametric approach. In that latter case, function g may no longer belong to one of the three basic families (for more details on methods to estimate parameters of Johnson distribution see Naguez and Prigent, 2010).

B.   Four First Moments

For the distribution S_U, the first four moments of the reduced and centered random Y are given by:

$$\mu_1 = -\omega^{\frac{1}{2}} sinh\Omega,$$

$$\mu_2 = \frac{1}{2}(\omega-1)(\omega\cosh 2\Omega + 1),$$

$$\mu_3 = -\frac{1}{4}\omega^{\frac{1}{2}}(\omega-1)^2\left[\omega(\omega+2)\sinh 3\Omega + 3\sinh\Omega\right],$$

$$\mu_4 = \frac{1}{8}(\omega-1)^2\left[\begin{array}{l}\omega^2(\omega^4+2\omega^3+3\omega^2-3)\cosh 4\Omega + \\ 4\omega^2(\omega+2)\cosh 2\Omega + 3(2\omega+1)\end{array}\right].$$

$$(1)$$

where $\omega = e^{\delta-2}, \Omega = \frac{\gamma}{\delta}.$

We note that the third and fourth moments depend on parameters γ and δ. In the financial context, this flexibility is interesting because this distribution can be used to model returns asymmetric or fat tail distributions. However, to model returns that follow Johnson distribution, we need to study how the moments of this distribution depends on its four parameters.

C.   Variations of the Four Moments w.r.t. the Parameters

In this section, we study the variations of the four first moments of the distribution S_U according to the four parameters.
We simulate returns that   follow a Johnson S_U distribution while changing parameter values for each simulation, in order to observe the variation of moments with respect to the variation of those parameters.
We study such modifications of moments according to the parameters since it allows us in a second step to study how the Kappa measures vary as function of the moments (via implicitly the parameters).
        Equations (1) show that the mean depends on all the parameters. The standard deviation depends on γ, δ, and λ. The skewness and kurtosis depend only on the parameters γ and δ. The sign of the skewness is the opposite of the sign of parameter γ.
        We also study the variations of the higher moments where the mean and the standard deviation are fixed.
        Figure 1 shows that skewness decreases with γ. It increases also with δ if γ is negative and increases if γ is positive. The kurtosis is an increasing function of δ and of negative values of γ. It is decreasing with respect to positive values of γ.
        These observations allow us to conclude that for given expected return and standard deviation, the two higher moments reach their respective maxima by minimizing γ and δ since they simultaneously provide the highest level of skewness and the highest level of kurtosis, a situation that does prefer the investor.

Figure 1

Variations of the skewness and kurtosis w.r.t. δ and γ
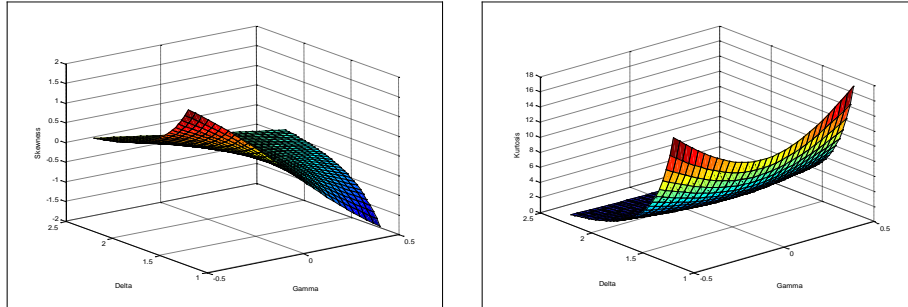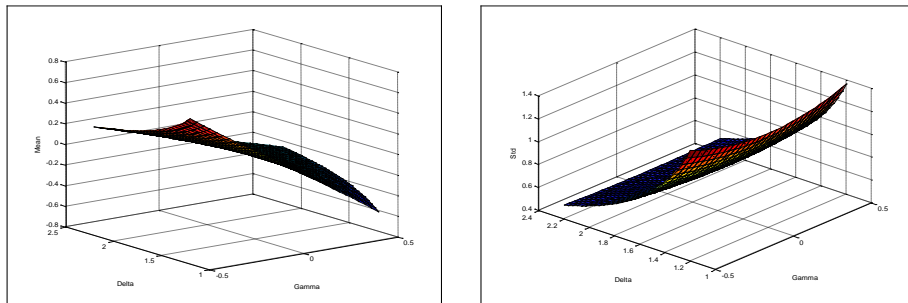

Figure 2

Variations of mean and standard deviation w.r.t. δ and γ

We also study the variations of the mean and standard deviation according to the parameters γ and δ with parameters ξ and λ fixed. Figure 2 shows that the mean is a decreasing function with respect to γ. It also decreases w.r.t. δ if γ is negative but increases if γ is positive. The standard deviation increases w.r.t. δ and positive values of γ but increases w.r.t. negative values of γ. From these observations, we can conclude that the worst case for an investor corresponds to simultaneous high value of γ and low value of δ.

IV.   KAPPPA MEASURES AND JOHNSON DISTRIBUTIONS

To study how Kappa performance measures vary according to the four moments of Johnson distribution, we have examined the variation of the first four moments within the Johnson parameters (see previous section). This allows illustrating how moments behave according to parameter variations. Then, in a second step, we study now how Kappa performance measures vary according to the Johnson parameters. Therefore, we deduce simultaneously how Kappa measures depend on the four first moments.

We consider that the return on the risky asset follows a Johnson distribution S_U. Therefore, Kappa performance measures are given by:

$$Kappa_l(L) = \frac{E_p\left[\lambda g^{-1}\left(\dfrac{z-\gamma}{\delta}\right)+\xi\right]-L}{\left(E_p\left[\left[\left(L-\lambda g^{-1}\left(\dfrac{z-\gamma}{\delta}\right)+\xi\right)^+\right]^l\right]\right)^{\frac{1}{l}}}$$

Note that closed-form representations of Omega with all Johnson type distributions (including the important bounded Johnson type) have been derived by Passow (2004). However, for more general Kappa measures, no explicit solution is known.

V.   NUMERICAL ILLUSTRATIONS

We study the variation of Kappa measures w.r.t. Johnson parameters which allows us to show the evolution of these measures w.r.t. the first four moments of this distribution. First, we study the variation of Kappa measures according to the four parameters of Johnson distribution. We find that Kappa measures are sensitive to all parameters. Referring to the results of Section III, we can observe that the four moments affect significantly the magnitude and the sense of variation measures Kappa, especially for lower values of the order k.

Then, we examine the variation of Kappa measures within the parameters δ and γ where the mean and standard deviations are fixed.

Figure 3
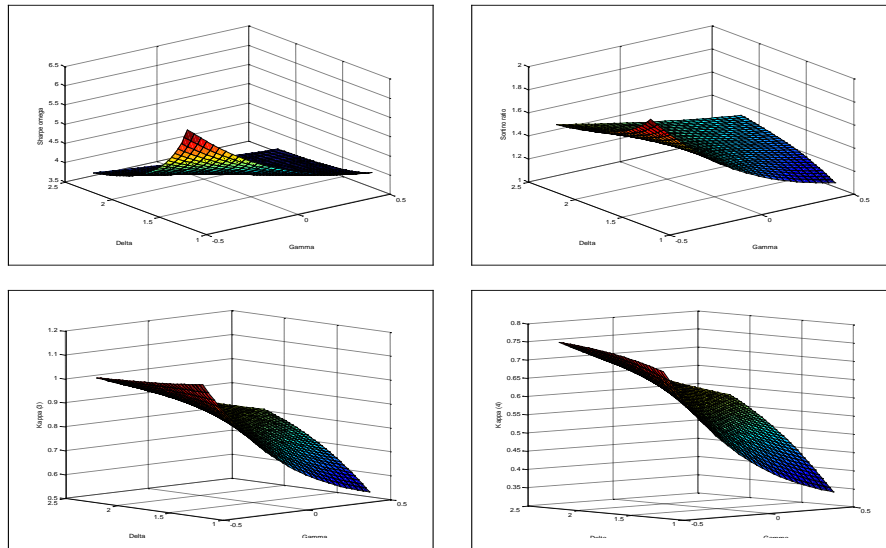Variations of Kappa for fixed mean and standard deviation

Figure 3 shows that all Kappa ratios decrease with γ. The evolution w.r.t parameter δ varies from one ratio to another. Indeed, the Sharpe Omega ratio is a decreasing function of δ, the Sortino ratio is an even function which grows to a certain level and then begins to decline.

The Kappa (3) ratio increases and then from a certain level begins to decrease, while the Kappa (4) ratio is an increasing function of δ. This implies that Kappa measures are increasing functions of the skewness whatever its sign and also increasing w.r.t kurtosis if the skewness is positive.

This is rational because the investment can achieve high performance if the distribution of returns favors the realization of positive returns with a high probability of very high returns.

We also study the variation of Kappa Measures w.r.t γ and δ for fixed ξ and λ. This allows us to determine the variation of Kappa w.r.t mean and standard variation.

Figure 4
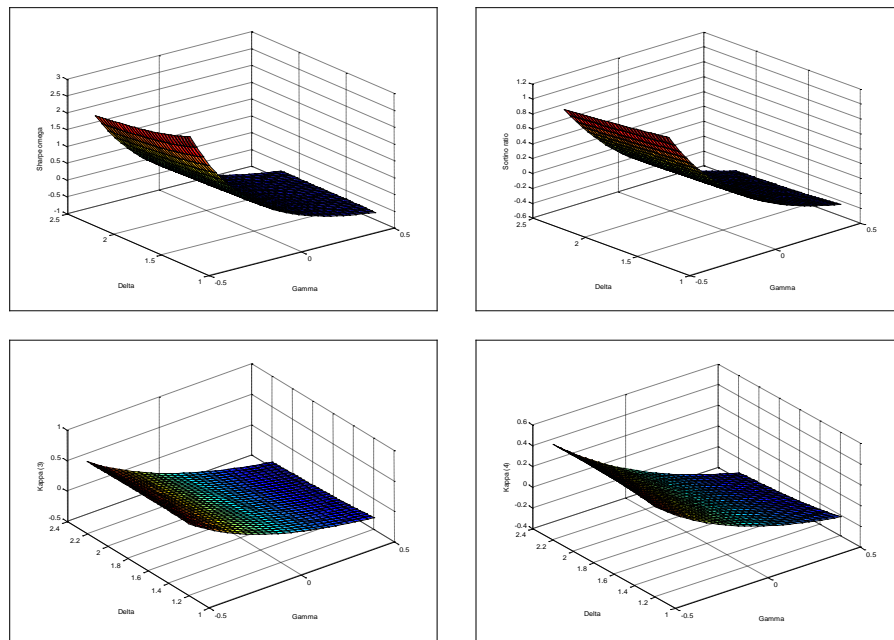Variations of Kappa for fixed ξ and λ

Figure 4 shows that Kappa measures are increasing functions w.r.t. γ and δ. This implies that the performance always increases with the mean and standard deviation if the skewness is positive but is decreasing with standard deviation if skewness is negative.

VI.   CONCLUSION

In this paper, we illustrate how the important family of Kappa performance measures behaves according to the four parameters of Johnson distributions. We also examine their variations with respect to their four moments. This allows the link of the maximization of Kappa performance measures with decision criteria based on the first four moments of the return probability distribution. This is particularly useful when dealing with non Gaussian distributions, as for hedge fund returns.

First, we show that Kappa measures always increase with the mean and standard deviation if the skewness is positive but are decreasing with standard deviation if skewness is negative; second we show that they are increasing functions of the skewness whatever its sign and also increasing w.r.t kurtosis if the skewness is positive. Therefore, for Johnson distributions, maximising Kappa measures would be prefer to maximize a combination of the first four moments since Kappa measures take better account of the "good" behaviour of the distribution (i.e. positive skewness).

ENDNOTES

1. Note that they are linked to the measures proposed by Fishburn (1977, 1984).
2. For example, interest rates, returns of benchmark financial indices, etc.
3. As noted by Kazemi et al. (2004), Omega can be considered as the ratio of the prices of a call option to a put option written on X with strike price L but both evaluated under the historical probability P.
4. X can denote log return depending on the type of Johnson distribution.

REFERENCES

Amenc, N., S. Curtis, and L. Martellini, 2002, "The Alpha and Omega of Hedge Fund Performance Measurement," *Working paper,* Edhec Risk and Asset Management Research Center.

Amenc, N., and L. Martellini, 2003, "The Brave New World of Hedge Funds Indices," *Working Paper,* Edhec Risk and Asset Management Research Center.

Artzner, P., Delbaen, F., Eber J.M., and D. Heath, 1999, "Coherent Measures of Risk," *Mathematical Finance* 9, 203-228.

Bacmann, J.F., and S. Scholz, 2003, "Alternative Performance Measures for Hedge Funds," *AIMA Journal*, June.

Bertand, P., and J.L. Prigent, 2011, "Omega Performance Measure and Portfolio Insurance," *Journal of Banking and Finance*, 35, 1811-1823.

Brooks, C., and H.M. Kat, 2001, "The Statistical Properties of Hedge Fund Index Returns and their Implications for Investors," *Working Paper,* ISMA Centre Discussion Papers In Finance.

Cascon, A., Keating, C., and W.F. Shadwick, 2003, "The Omega Function," *Working paper,* The Finance Development Centre London.

Farinelli, S., and L. Tibiletti, 2008, "Sharpe Thinking in Asset Ranking with One-sided Measures," *European Journal of Operational Research* 185, 1542-1547.

Fishburn, P.C., 1977. "Mean-risk Analysis with Risk Associated with Below-Target Returns," *American Economic Review* 67, 116-126.

Fishburn, P.C., 1984, "Foundations of Risk Measurement. I. Risk as Probable Loss," *Management Science* 30, 396-406.

Fung, W., and D. Hsieh, 1997, "Empirical Characteristics of Dynamic Trading Strategies: The Case of Hedge Funds," *Review of Financial Studies* 10, 275-302.

Jarrow, R., and F. Zhao, 2006, "Downside Loss Aversion and Portfolio Management," *Management Science* 52, 558-566.

Johnson, N.L., 1949, "Systems of Frequency Curves Generated by Methods of Translation," *Biometrica*, 58, 547-558.

Kaplan, P. and J.A. Knowles, 2004, "Kappa: A Generalized Downside Risk-adjusted Performance Measure," *Journal of Performance Measurement* 8, 42-54.

Kazemi, H., T. Schneeweis, and R. Gupta, 2004, "Omega as Performance Measure," *Journal of Performance Measurement*, 8, 16-25.

Keating, C., and W.F. Shadwick, 2002, "A Universal Performance Measure," *Journal of Performance Measurement* 6, 59-84.

Naguez, N., and J.L. Prigent, 2010, "Arch Models with Johnson Distributions as Innovations," *Working paper,* THEMA.

Passow, A., 2004, "Omega Portfolio Construction with Johnson Distributions," *FAME Research Paper N° 120, HEC Geneva.*

Pedersen, C.S., and S.E. Satchell, 1998, "An Extended Family of Financial-risk Measures," *The Geneva Papers on Risk and Insurance Theory* 23, 89-117.

Pedersen, C.S., and S.E., Satchell, 2002, "On the Foundation of Performance Measures under Asymmetric returns," *Quantitative Finance,* 2, 217-223.

Szegö, G., 2002, "Measures of Risk," *Journal of Banking and Finance,* 26, 1253-1272.

Tversky, A., and D., Kahneman, 1992, "Advances in Prospect Theory: Cumulative Representation of Uncertainty," *Journal of Risk and Uncertainty* 5, 297-323.

Unser, M., 2000, "Lower Partial Moments as Measures of Perceived Risk: An experimental study," *Journal of Economic Psychology* 21, 253-280.

Zakamouline, V., 2010, "Portfolio Performance Evaluation with Loss Aversion," *Working paper, University of Agder, Faculty of Economics, Norway.*

Zakamouline, V., and S. Koekebakker, 2009, "Portfolio Performance Evaluation with Generalized Sharpe Ratio: Beyond the Mean and Variance," *Journal of Banking and Finance* 33, 1242-1254.