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Fuzzy Asset Pricing Under the Law of One Price

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ABSTRACT

With the law of one price, a fuzzy asset pricing provides an alternative for asset/property evaluation. We create two pseudo assets to mimic the asset being evaluated. Pseudo assets are constructed by two sets of comparable assets within the same industry/sector and then to solve a unique solution of systematic simultaneous equations for each set, such that each pseudo asset factors' components are identical to the asset being evaluated. Then we calculate both pseudo asset pricings according to their solution based on the simultaneous equations detailed below. With the pseudo prices, we set a fuzzy asset pricing range for the asset under the law of one price. This fuzzy pricing model serves two purposes: (2) estimating asset pricing within this fuzzy range; and (2) using the fuzzy range as a yardstick for evaluating the realized performance of any asset class, such as equity, property, or mutual funds.

JEL Classification: C6,C8,G0, G1,G5,R0,R3

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I. INTRODUCTION

Those who have been critical of the singular CAPM have proposed a variety of valuation alternatives. The most common alternative valuation models, namely the spinoffs of CAPM, depend on beta factors and are point estimates. Besides the single point estimate of asset valuation, most previous asset pricing models are linearly related to the sensitivities of the factors.[1] In fact, although the factors are observable (or estimable), the real asset value is unknown and thus must be estimated. In addition, the asset value it may not necessary be linearly related to its factors or their betas. Hence, the asset pricing model based upon linearity is presumed without justification. Furthermore, to comply with the law of one price, the different asset's component factors imply different asset values, indicating that an arbitrage opportunity exists.

By imposing the law of one price and employing multiple factors, this paper provides a fuzzy asset pricing range for the performance of an equity, property or mutual fund in the same industry or sector, rather than the point (or single) estimate, e.g., previous models like CAPM, ICAPM, APT, and multiple factor models. Our valuation technique is a type of comparable analysis and can serve to estimate an unknown asset's value or to examine the performance of equity, property, …, etc. under the law of one price; in terms of whether the performance is superior or inferior when compared to normal performance. For example, our technique can be used to examine a mutual fund's realized performance at the end of a period.

The most frequent model for evaluating equity in finance is CAPM and its derivates, in which the market risk (beta) is the only or crucial factor for asset pricing. However, CAPM was claimed dead by Lai and Stohs (2015), among others in a variety of respects going back at least to Fama and French (1996). It is dead because CAPM is a tautology and is useless for asset pricing in terms of logic and statistics. Lai and Stohs (2021) further prove mathematically that CAPM is exactly the first order condition for the optimal portfolio, a simultaneous linear equations, with the optimal risky portfolio solution under the Kuhn–Tucker condition within the mean–variance framework.[2] The first order condition with its solution of an optimal risky portfolio is, of course, a tautology, otherwise the portfolio would not be the unique solution for that condition for optimality. The first order condition is purposely used to solve for an optimal risky portfolio and is inappropriate as a foundation for a pricing model that postulates a linear function between beta and its expected rate of return on an asset, because simultaneous equations are not mathematical functions. Therefore, the market portfolio is either identical to the unique optimal risky portfolio for the first order condition and thus the CAPM results in a tautology and is useless for asset pricing, or the first order condition (i.e., CAPM) fails.

Factor models comprise other common asset valuations, usually as spinoffs of CAPM. They explore common factors affecting the assets' premiums in the market. The Arbitrage Pricing Theory Model (hereafter APT) developed by Ross (1976) assumes k common factors, instead of a single factor of market risk (or beta) used in CAPM to evaluate the asset's premium under the condition of no arbitrage opportunity. APT asserts that the expected excess rate of return on asset linearly depends on the sensitivities (betas) to the factors. APT is one of the predominant applications of the factor models. Chen et al. (1986) maintain that four macroeconomic variables are significant for security returns.[3] In addition, a wealth of studies examine multi–factor models, some using

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different macroeconomic factors to estimate the asset premium in the market. Multi–factor asset pricing models extend from the single beta CAPM, with APT serving as the primary theoretical underpinning of multifactor models. Fama and French (1993) propose the Three–Factor Model, which is a multiple regression model. Carhart (1997) adds a fourth factor of momentum.[4] Fama and French (2015) extend their three factors to add two more factors – Profitability (RMW) and Investment (CMA) to become a five–factor model. It uses the return of stocks with high operating profitability minus the return of stocks with low or negative operating profitability. Examples of such factors are price–to–earnings ratio, market capitalization, and financial leverage.

Despite the widespread use of multi–factor models, they all rely on the basic misspecification of CAPM, and thus they share the technical obstacles inherent to the original single–beta CAPM.[5] All previous factor models assume that securities' returns are a linear function of common factors (or betas of factors), which affect all assets' values in the market. Individual asset returns are estimated through the multiple regressions. No consensus exists about the exact nature and number of the common factors that do or should exist for asset pricing. Further, individual market sectors may have significant unique features (or factors) that differentiate their assets and values. For example, the factors for the utility industry should differ from that of the high technology industry. Hence, it may be more prudent to identify factors common to a given industry that affect the assets or products in that industry, than to rely solely on macroeconomic factors that affect all assets in the market. This paper explores this latter approach, and in so doing, we assume that the common factors are given, as in prior asset pricing models.

In contrast to the multi–factor models, we do not require linearity between an asset's value and its factors' returns (or betas of factors). Simply, the law of one price and the elements of each factor, and other comparable assets' realized values as used in multiple regression in the same sector/industry, are sufficient to construct a firm–specific evaluation range (a fuzzy range) in this paper. If the current market price of an asset falls within this fuzzy range, we designate that evaluation itself (for that firm) as normal, otherwise is abnormal. In addition, although some macroeconomic factors are not explicitly included in the evaluation, the effects of all macroeconomic factors would be incorporated in other comparable assets' values. As a result, the missed macroeconomic effects are implicitly contained in the valuation through setting the upper and lower bound of the fuzzy pricing range (see Equations (3) and (6)), even with those factors being explicitly excluded in that asset's evaluation. The purpose of this paper is to identify comparable assets within a given industry, rather than to provide an asset pricing model for all assets in the market. Hence, it is an open question whether any macroeconomic factors would be included in any given application of the fuzzy asset pricing model. In addition, there is no suggestion that there will be a unique set of factors in any given application of the fuzzy asset pricing model.

Under these circumstances, we separate the comparable assets within the same industry into two sets, with each set having the same number of factors. One set includes the highest asset realized values and the other includes the lowest realized values. We create a pseudo asset from each set to mimic the being evaluated asset by solving the simultaneous equations that forces the components (ingredients) for the pseudo asset to be identical to the components of the asset being evaluated. Since there are two sets of comparable assets, two solutions result. We presume that comparable assets' components of factors have already been identified, like multiple regression, and all the comparable

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assets' realized value are ranked. By analogy, simply consider ranking the realized comparable assets' realized values in a group according to their realized (or estimated) values (i.e., the goal of evaluation such as the rate of return, mutual funds' performances, sold prices of property, etc.).

Subsequently, we use these two solutions to calculate the pseudo asset pricing. The two pseudo pricings allow us to set a range of upper and low bounds for the fuzzy pricing of that asset. The evaluation will be within the fuzzy range, and thus within a second sense "normal," for being evaluated asset or to examine the performance if it over– or under–performs and thus be a normal or abnormal performance. The outcome is a fuzzy evaluation range rather than a point estimate for the being evaluated, or to evaluate whether the performance is normal or not, if the number of factors k less than that of the comparable assets n. A normal estimate of the asset falls within the range. Evaluations outside the range indicate abnormal performance. In other words, it would be superior if the estimate exceeds the upper bound, and inferior if the performance is lower than the lower bound.

It is true that common factors within an industry may affect the risk premiums of firms differently. And in some respects, any individual firm or asset has its own unique factors. However, the fuzzy asset pricing model results in a justifiable price range, rather than a specific (point) value for an asset. Thus, there is no claim that the fuzzy asset pricing model provides a unique value for an asset, and no claim that the unique common factors for an industry are necessary for a valid valuation range. Notice that prior asset pricing models also utilize factors common to all assets. There is an important way in which issues related to identifying common factors automatically preclude unique pricing for an asset. The fuzzy asset pricing model is simply more explicit in recognizing this fact.

II. FUZZY ASSET PRICING UNDER THE LAW OF ONE PRICE

We assume that an asset's value depends on its components of factors. In residential housing, for example, price depends on the number of bedrooms, square feet of living, lot size, number of baths and rooms, all of which are the factors affecting the home's value in the housing sector. We assume there are k factors in the sector that affect the asset valuation (or performance of mutual funds manager) $F_1, F_2, \cdots, F_k$, and there are n given comparable assets pricings $Y_1, Y_2, \cdots, Y_n$, with $n > k$. Of course, different sectors have different number of k factors and comparable assets. The numbers of k, n, and factors depend on the sectors. The term a_{ij} stands for the component of the i^{th} asset with j^{th} factor, $j = 1,2, \ldots, k$. In other words, the value of i^{th} asset depends on its components (ingredients) of k–factors $a_{i1}, a_{i2}, \cdots, a_{ik}$, $i = 1,2, \cdots, n$. Although there is no functional relationship between the asset's value and the factors, e.g., no presumed linearity or any other functional form, the relationship between the asset value Y and its component a_{ij} for each factor is represented in Table 1, where Y_b is either the unknown asset value will be evaluated (or estimated) or the performance to be examined (or compared with others), and 1xk row vector $\mathbf{b} = (b_1, b_3, \ldots b_k)$, b_j, $j = 1, 2, \cdots, k$ are its components of factors, $k = 1, 2, \ldots, k$. According to the performance, we assume comparable assets' realized values Y_i, $i = 1, 2, \cdots, n$ are listed in decreasing order (i.e., the top is the highest realized value).

Table 1
Matrix of Asset Values and Factors

Values	F_1	F_2	…	F_k
Y_1	a_{11}	a_{12}	…	a_{1k}
Y_2	a_{21}	a_{22}	…	a_{2k}
⋮	⋮	⋮	⋮	⋮
Y_k	a_{k1}	a_{k2}	…	a_{kk}
⋮	⋮	⋮	…	⋮
Y_{n-k+1}	a_{n-k+11}	a_{n-k+12}	…	a_{n-k+1k}
⋮	⋮	⋮	⋮	⋮
Y_n	a_{n1}	a_{n2}	…	a_{nk}
Y_b	b_1	b_2	…	b_k

Table 1 demonstrates the relationships between the value of asset Y and its components, a_{ij} for each factor. The Factors range from 1 through k, analogous, though not identical, to the factors in a standard K–factor asset pricing model. The term a_{ij} stands for the component of the i^{th} asset with j^{th} factor. In other words, the value of i^{th} asset depends on its components (ingredients) of k–factors $a_{i1}, a_{i2}, \cdots, a_{ik}$, i= 1,2, ⋯, n.

We create two pseudo assets/properties to mimic their factors' components which are identical to that of the asset being evaluated. Regarding **Table 1**, we take the first k maximum Y_i, i = 1, 2, ⋯, k, and last k minimum Y_i, i = n-k+1, n-k+2, ⋯, n with the components of $\mathbf{A}^u$ and $\mathbf{A}^d$, respectively, where $\mathbf{A}^u$ and $\mathbf{A}^d$ are:[6]

$$\mathbf{A}^d = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} \quad , \qquad \mathbf{A}^d = \begin{bmatrix} a_{n-k+11} & \cdots & a_{n-k+1k} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{bmatrix}$$

respectively. $\mathbf{A}^u$ and $\mathbf{A}^d$ are assumed convertible k x k matrices. To comply with the law of one price, we use the comparable assets' components (ingredients), $\mathbf{A}^u$, with highest pricings to construct the pseudo asset with identical ingredients **b** to that of the asset being evaluated is to solve the variables of 1 x k row vector $\boldsymbol{x}^{u'} = (x_1, x_2, \cdots, x_k)$ for the following simultaneous equations such that

$$\mathbf{x}^{u'}\mathbf{A}^u = \mathbf{b} \qquad (1)$$

Rewriting, we solve the row vector $\boldsymbol{x}^{u'} = (x_1, x_2, \cdots, x_k)$ such that

$$(x_1, x_2, \cdots, x_k) \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{bmatrix} = (b_1, b_2, \cdots, b_k) = \mathbf{b} \qquad (1)$$

in which **b** is a 1 x k row vector of b_j which is being evaluated component of asset's factors F_j, j =1, 2, ⋯, k.[7] In detail, Equation (1) can be rewritten as (1a) to find the solution $\boldsymbol{x}^{*u}$ of the following simultaneous equations of $\boldsymbol{x}^u$ to mimics the components **b** of the asset being evaluated:

$$\begin{aligned} a_{11}x_1^u + a_{21}x_2^u + \cdots + a_{k1}x_k^u &= b_1 \\ a_{12}x_1^u + a_{22}x_2^u + \cdots + a_{k2}x_k^u &= b_2 \\ &\vdots \end{aligned} \qquad (1a)$$

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$$a_{1k}x_1^u + a_{2k}x_2^u + \cdots + a_{kk}x_k^u = b_k$$

In these equations, the variable $\boldsymbol{x}^{u\prime} = (x_1^u, x_2^u, \cdots, x_k^u)$ is the transpose vector of $\boldsymbol{x}^u$, as explained above. Equation (1) is a simultaneous linear equation with k equations and k variables of $x_1^u, x_2^u, \cdots, x_k^u$ and serves as the constraints that the pseudo asset's ingredients of factors are identical to that of the asset being evaluated to fulfil the law of one price.

The unique solution 1 x k vector $\boldsymbol{x}^{*u\prime}$ of the simultaneous Equation (1) is

$$\boldsymbol{x}^{*u\prime} = \mathbf{b}(\mathbf{A}^u)^{-1} \quad (2)$$

According to the solution of the simultaneous equation, the pseudo asset is constructed by the top k assets with the weight of x_j^{*u} to the j^th^ asset, j=1, 2, ⋯, k. Equation (1) shows that the pseudo asset has the identical components **b** to the being evaluated asset. The pseudo asset's pricing Y^{*u} is thus calculated by

$$Y^{*u} = \boldsymbol{x}^{*u\prime}\mathbf{Y}^u = \mathbf{b}(\mathbf{A}^u)^{-1}\mathbf{Y}^u = x_1^{*u}Y_1 + x_2^{*u}Y_2 + \cdots + x_k^{*u}Y_k \quad (3)$$

Here $\mathbf{Y}^u$ is k x 1 vector, the transpose of 1 x k vector $\mathbf{Y}^{u\prime} = (Y_1, Y_2, \cdots, Y_k)$. Y^{*u} is a 1 x 1 scalar and will be used as the upper bound of the fuzzy range for asset pricing with its factor components of **b**. Since this pseudo asset is constructed by the highest top k assets pricings under the constraints of **b**, Y^{*u} must be the maximum pricing with components **b** and thus Equation (3) can be used as the upper bound of fuzzy range for evaluation with the same components of **b**.

Similarly, by using the bottom k comparable assets or properties from Table I, we can obtain the lower bound of fuzzy range for asset the evaluation by constructing the pseudo asset with identical components **b** to that of being evaluated through solving ~~e~~ $\boldsymbol{x}^{d\prime} = (x_1^d, x_2^d, \cdots, x_k^d)$ for the following simultaneous equations such that

$$\boldsymbol{x}^{d\prime}\mathbf{A}^d = \mathbf{b} \quad (4)$$

In detail, (4) can be rewritten as (4a):

$$\begin{aligned} a_{n-k+11}x_1^d + a_{n-k+21}x_2^d + \cdots + a_{n1}x_k^d &= b_1 \\ a_{n-k+12}x_1^d + a_{n-k+22}x_2^d + \cdots + a_{n2}x_k^d &= b_2 \\ &\vdots \\ a_{n-k+1k}x_1^d + a_{n-k+2k}x_2^d + \cdots + a_{nk}x_k^d &= b_k \end{aligned} \quad (4a)$$

as explained above. The solution $\boldsymbol{x}^{*d\prime} = (x_1^{*d}, x_2^{*d}, \cdots, x_k^{*d})$ of Equation (4) is

$$\boldsymbol{x}^{*d\prime} = \mathbf{b}(\mathbf{A}^d)^{-1} \quad (5)$$

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The lower bound of pseudo asset pricing Y^{*d} is decided by multiplying the solution $\boldsymbol{x}^{*d}$ from Equation (5) and the corresponding the bottom performance 1xk row vector $\mathbf{Y}^{d'} = (Y_{n-k+1}, Y_{n-k+2}, \cdots, Y_n)$ from Table I. That is,

$$Y^{*d} = \boldsymbol{x}^{*d'}\mathbf{Y}^{d} = \mathbf{b}(\mathbf{A}^{d})^{-1}\ \mathbf{Y}^{d} = x_1^{*d}Y_{n-k+1} + x_2^{*d}Y_{n-k+2} + \cdots + x_k^{*d}Y_n \tag{6}$$

Equations (3) and (6) are calculated from the k maximum and k minimum asset pricings with the identical components **b**, hence, the fuzzy range to estimate for asset Y_b should be located between the range of Y^{*d} and Y^{*u} :

$$Y^{*d} \leq Y_b \leq Y^{*u} \tag{7}$$

Equation (7) should hold for all assets with the same components of factors **b** because Equations (3) and (6) are constructed, respectively, by the maximum and minimum comparable asset pricings for both pseudo asset pricings in the same industry. If the realized asset pricing with the components **b** locates outside of the fuzzy range of Equation (7), then the performance is either superior or inferior to the normal.

III. ADVANTAGES AND DISADVANTAGES OF A FUZZY ASSET PRICING MODEL

Previous studies demonstrate the difficultly of deciding how many macro–factors should be used for all assets on the market. At this point in time, any interpretation of the meaning of the factors appears subjective, without any consensus in the literature. Selecting a good set of factors is complicated, and different researchers will choose different sets of factors for all assets in the market. Since the whole market consists of all industries, any given set of factors may not be appropriate for all industries. It may not be easy to get a consensus for common factors which affect all assets in the market. Instead of relying on the whole market, we focus on an industry or sector of assets which should allow researchers to reach a consensus concerning a good set of factors common to all assets values in the given industry, according to market segmentation. In addition, the fuzzy pricing model provides arrange for estimation of asset against the point estimation, the probability of its accuracy is zero, so the fuzzy asset pricing model is more plausible than previous asset pricing models, at least in theory.[8]

Any force (or effects) caused by common macroeconomic factors on an asset's value would be incorporated into the individual asset value. In this paper, through the construction of the pseudo asset, the effect of that macroeconomic factor will be integrated into the fuzzy range, even though some factors are not explicitly incorporated into the construction of the valuation. In addition, since our alternative valuation does not require the assumption of a functional form for the evaluation, it remains free from the linear or other function forms requirement embedded in multi–factor models to circumvent the misspecification. The fuzzy pricing model requires no need to use regressions or any statistic calculation for estimating the sensitivities (i.e., coefficients or betas) of factors, although linear algebra is required for mimicking the pseudo assets' components to comply the law of one price. The disadvantage in our innovative approach is this is the firm–specific model, which is unique for a firm/property. Since different firms/properties differ in their components of factors, the fuzzy range for each

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firm/property will be different to comply according to the law of one price. Therefore, the fuzzy range for one asset's evaluation may not apply to other assets in the same industry if the components of factors are not the same.

In addition, multi-factor evaluation requires sensitivities (betas) of factors for the expected rate of return. Due to lack of the density function for asset returns, estimating the betas of factors may not easy. Hence, the beta for a factor is likely to be an empirical result by using regression based on historical time series rather than contemporary data. In contrast, the components of fuzzy pricing are almost intertemporally obtainable for the study. The components of the factors and the comparable assets are the only information needed comply the law of one price. In other words, solving the simultaneous linear equations with certainty/observable coefficients (i.e., components of comparable assets) of factors to mimic those of the asset being examined should be easier than to estimate or calculate the betas of factors. Furthermore, this fuzzy evaluation is free from the sensitivities and does not require a regression for estimation. Therefore, this fuzzy evaluation is free from the regression problems inherent in APT or multi-factor models.

IV. SPECIAL SCENARIOS

What if a comparable asset's ingredients or components are identical to those of the asset being evaluated in the same industry? To prevent the presence of an arbitrage opportunity in this scenario, one can use this comparable asset price as the price of the asset being evaluated. An alternative is we can exclude this comparable asset and add another asset in the fuzzy evaluation construction if $n > k$. Previous construction of the pseudo assets to mimic the asset being asset and evaluations can still be completed with that replacement. After the replacement, if that comparable assert pricing is out of the fuzzy range, then the fuzzy range should be revised by using the realized value of the comparable asset for the upper or lower bound, since this comparable asset meets the requirement of the law of one price.

The fuzzy range plays the role as a reference for asset pricing, and is appropriate for cases such as the presence of patents, fake assets, potential lawsuits, …, etc. Thus, the resulting fuzzy set from this modification holds for the purpose of fuzzy evaluation. In addition, what procedure should be used when the number of assets is same as the number of factors? We suggest either reverting to a point estimation as discussed by Lai's et al. (2008) research about property values, in instead of a fuzzy evaluation, or else add (reduce) the number of comparable assets n (factors k) so that range of valuations is mathematically possible.

V. CONCLUSION

This paper explores a fuzzy pricing model; an alternative asset/property evaluation pricing model, to circumvent the controversies of macroeconomic factors and the factors used in multi–factor models of asset evaluation in the market. The consensus of number and common factors in the same industries/location should be easier than the current market models. With the law of one price, we create two pseudo assets and set a fuzzy pricing range to estimate or examine that asset/property value rather than a point estimation. This alternative evaluation is free of the functional form (e.g., linear relationship, which is unlikely the true functional form) of asset's value with its factors,

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only the components of factors and comparable assets values in the same industry are enough for the fuzzy evaluation for that asset.

As for all valuation techniques, the true asset value remains unknown, and there is no alternative but to use valuation estimates. The alternative method presented here should be used for examining an individual asset, such as residential real estate, or individual mutual fund managers' performance. It should be apparent that evaluation by comparables is standard in asset evaluation, the housing industry, managers' evaluation at the end of period, … , etc., and has the same basis of the law of one price.

Finally, this fuzzy pricing model can apply in the practice because it provides a flexibility of providing a range for asset prices, in which the buyer and seller can facilitate arriving at a rational transaction (e.g., merger a firm, counteroffer in real estate, …, etc.). The result should be that both sides are satisfied with the transaction, because the buyer did not pay at the highest range nor did the seller sell at the lowest range. We add to general valuation techniques by providing a mathematically sound foundation for such comparable analysis.

By imposing the law of one price and employing multiple factors, this paper provides a fuzzy asset pricing range for the performance of an equity, property or mutual fund in the same industry or sector, rather than the point (or single) estimate, e.g., previous models like CAPM, ICAPM, APT, and multiple factor models. Our valuation technique is a type of comparable analysis and can serve to estimate an unknown asset's value or to examine the performance of equity, property, …, etc. under the law of one price; in terms of whether the performance is superior or inferior when compared to normal performance. For example, our technique can be used to examine a mutual fund's realized performance at the end of a period.

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ENDNOTES

[1] In Statistics, the expectation of a random variable is calculated from ~~the~~ its density function. I.e., the probability of random variable x equals to a fixed number a is Pro(x = a) $=\int_a^a f(x)\,dx = F(a)-F(a) = 0$, where f(x) is the density function of x, F(x) is the distribution of x, its differential dF(x) = f(x)dx, and a is a fixed number.

[2] With the Kuhn-Tucker condition and the assumptions of homogeneous belief, the unique optimal risky portfolio must be identical for all investors in the market within the mean-variance framework. The optimal risky portfolio for all investors implies that the market portfolio (through the aggregation of all investors' optimal risky portfolios) in the CAPM must be identical to the unique optimal risky portfolio otherwise the first order condition would be violated, so is the CAPM. Thus, the CAPM (i.e., the first order condition with the optimal risky solution) inevitably becomes a tautology rather than an asset pricing model.

[3] They are Industrial Production, Spread Between High- and Low-Grade Bonds, Spread between Long-term and Short-Term Interest Rate, and Unexpected Inflation Rate.

[4] The three factors include the SMB (small minus big), HML (high minus low), and the portfolio return minus the risk-free rate. SMB characterizes publicly traded companies with small market caps that generate higher returns, and HML uses value stocks with high book-to-market ratios that generate higher returns relative to the market.

[5] In fact, as argued by Lai and Stohs (2015), using the expected rate of return as the dependent variable and beta (market risk), which depends on optimal risky portfolio expected rate of return, as the independent variable in CAPM is a statistical misspecification. This procedure is illogical because the expected rate of return exists logically prior to the calculation of market risk (beta).

[6] Boldface indicates a vector or matrix in this paper.

[7] In k-dimension Euclidean spaces, any k-element vector can be generated by k independent k-element vectors.

[8] The Probability of a continuous random variable x between a and b is Pro(a < x < b) $=\int_a^b f(x)dx = F(b) - F(a) > 0$, if b > a, where F(x) is the density function of x and F(x) is its distribution, i.e., dF(x) =f(x)dx.

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