INTERNATIONAL JOURNAL OF BUSINESS, 26(4), 2021 ISSN: 1083-4346

CAPM and Asset Pricing

Tsong-Yue Lai[a] and Mark Hoven Stohs[b]

[a]Emeritus Professor of Finance
California State University Fullerton
tylai@fullerton.edu
[b]Professor of Finance
Department of Finance
California State University Fullerton
mstohs@fullerton.edu

ABSTRACT

We review and investigate the difficulties with CAPM as a foundation for asset pricing by analyzing the proof that CAPM is the necessary condition for optimizing the objective function of an optimal portfolio within the mean–variance framework and with the Kuhn-Tucker conditions. Though the market portfolio in CAPM is the unique solution of the necessary condition, the optimal portfolio of risky assets or the necessary condition provides no basis for individual expected rates of return for those assets. This is why CAPM is dead for asset pricing. In addition, the necessary condition is a simultaneous linear equation rather than a linear function, i.e., the presumed linearity between the beta and expected rate of return on assets does not exist. Therefore, all previous empirical findings based on CAPM are dubious at best, and should be reexamined by other methods. Empirical testing of CAPM does not establish its veracity, i.e., CAPM is vacuous as a pricing model.

JEL Classifications: G1,G10, G12

Keywords: beta, CAPM, kuhn-tucker conditions, market portfolio, mean–variance efficient portfolio

I. INTRODUCTION

We explore why the capital asset pricing model (CAPM) is dead for asset pricing by re-examining the original derivation of CAPM, noting that it follows from the **first order condition[1] (FOC)** within the mean–variance framework. That is, CAPM is the FOC, if and only if the market portfolio is the unique solution of FOC. The FOC, which enables us to obtain the unique optimal portfolio, requires a solution to optimize the objective function of portfolio mean and variance. The FOC[2] is unrelated to asset pricing, which focuses instead on estimating the cost of equity for individual assets. We argue that CAPM provides no basis for asset pricing, despite the strong inducement to do so by its derivation and the nature of the FOC, and should not be used to obtain a parameter like an expected rate of return.

Though objective functions may differ, the FOC (a simultaneous equation) together with the Kuhn–Tucker conditions yield a convergence to the same optimal portfolio within the mean-variance framework. For example, Sharpe[3] (1964), Mossin[4] (1966), and Merton[5] (1973) use the FOC for the optimal portfolio to maximize the utility function in their derivations of CAPM, while Lintner[6] (1965) obtains the FOC for maximizing the Sharpe ratio as the objective function to find the optimal portfolio in his derivation of CAPM. Other derivations use the FOC condition to find the optimal portfolio for the objective function, and then set (or aggregate) the optimal portfolio to the market portfolio to derive the CAPM in market equilibrium.

Despite this previous research, we focus on the fact that the derivation of CAPM using the FOC generates a unique optimal portfolio which is the central to CAPM. Since the FOC cannot serve as the basis for asset pricing, what we traditionally label CAPM plays no role for asset pricing, whether or not the market portfolio is efficient.

In addition, based upon the mathematical programming of minimizing portfolio risk subject to the constraint of an expected rate of return, according to the FOC of the Lagrangian, Roll (1977) concludes that 'there is an "if and only if" [necessary and sufficient condition] relation between return/beta linearity and market portfolio mean–variance efficiency.' In other words, any valid use or test of CAPM 'presupposes complete knowledge of the true market portfolio's composition.' [7]CAPM is an asset pricing model for individual assets, **if and only if** we have truly identified the market portfolio's composition.[8] Unfortunately, as proved in this paper, the CAPM is not an asset pricing model, whether or not the market portfolio's composition can be truly identified.

Further, the return/beta relationship is a solution of a simultaneous system of linear equations with n variables and n linear equations. It is the necessary condition with the Kuhn-Tucker conditions, rather than a linear function. This paper demonstrates that the FOC with the mean-variance efficient portfolio (i.e., the CAPM) is a system of simultaneous equations, which is used to solve the optimal solution, rather than being used to conclude the linearity between return and beta. The linearity of expected rate of return and risk, as stated in previous research, does not exist because the expected excess rates of return are parameters, rather than dependent variables. Essentially, as shown in this paper, the beta in CAPM is calculated from the mean-variance portfolio which must depend on the expected rate of return. Thus, beta depends on expected rate of return not vice versa. In fact, according to the FOC of optimality, the utility function of portfolio mean and portfolio risk (the linear equations in the FOC) implies that a higher expected excess rate of return results in greater weight, and thus generates a higher beta. This can be examined by comparative statics analysis. For example, if the FOC is ax=b, given a >0 , with the solution x*= b/a, comparative statics analysis shows $(\partial x^*)/(\partial b)=1/a > 0$, with a higher b, the solution x* (i.e., with x as the decision variable) must be higher. For a simple numerical example, given 2x = 3 and increasing 3 to 5 to become 2x = 5, the solution must increase from 3/2 to 5/2. This shows that greater the b, the greater the solution of x* for the FOC of ax = b; the converse of the CAPM claim that the higher the risk, the higher the expected rate of return. The claim of the CAPM is reversed.

We show that even if the market portfolio is mean–variance efficient and all other variables are available and accurate, CAPM still cannot serve as a basis for asset pricing. Because the market portfolio used in practice is just a portfolio of assets rather than a portfolio constructed by mean-variance (i.e., it is not the solution of the FOC), it cannot provide guidance for estimating individual expected rates of return. Should an expected rate of return for a given asset be derivable from the market portfolio, then the market portfolio would explain the asset's expected rate of return; yet the aggregate (market) expected rate of return is explained by the rates of return of the individual assets. This problem is either a tautology or circular reasoning. In statistical terms, it is a serious case of endogeneity which cannot be eliminated in practice.[9] Minimizing the problem of endogeneity is perhaps theoretically possible, but only in an ideal world in which the market portfolio consists of ALL marketable assets, perfect competition exists among all assets, and the portfolio is mean–variance efficient. In this ideal world, the impact of any one asset's expected return on the expected return of the market, in the limit, would tend towards zero, thus eliminating or minimizing the endogenous nature of the basic problem.

After nearly six decades of persistent research, the theoretical and empirical controversies surrounding CAPM remain. While merely indicative of research interest, a quick Google Scholar search for CAPM within finance results in nearly 100,000 results. Given this plethora of research items, consider just a few. For example, CAPM is supported empirically by Black, Jensen, and Scholes (1972) and Fama and MacBeth (1973). Jungshik and Kumar (2007) use their empirical results to claim that beta may not be dead after all. Fama and French (1996) use the title of "The CAPM is Wanted, Dead or Live" for their paper to challenge the CAPM's validity. Jagannathan and Wang (1994) claim the CAPM is alive and well. Levy (2010) claims CAPM is theoretically valid even when one accepts the BE&P (behavioral economists and psychologists) framework and expected utility is invalid. Lai and Stohs (2015) use algebraic analysis[10] (i.e., the linear transformation of covariance matrix between the portfolio to the expected excess rates of return on assets and the portfolio) and statistics (e.g., the mean must exist prior to the calculation of the covariance) to prove that CAPM is dead for asset pricing.

Many studies claim it is too early to conclude that CAPM is dead. For example, the title used by Kothari, Shanken and Sloan, Richard G (1998) is "The CAPM: 'Reports of My Death Have Been Greatly Exaggerated'." Many empirical studies attribute the failure of CAPM to suspect data, such as inadequate proxies, risk free rate, data mining, etc. For example, Harvey et al (2016) direct suspicion against most empirical research in financial economics concerning the statistical significance of many past studies. Disputes about the asset–pricing uses of CAPM have never ended, and most likely never will. We argue that those disputes should end because debate about a tautology CAPM is meaningless.

One suggestion for why such disputes won't end stems from the basic two–step process for deriving CAPM and the subsequent SML. The math of the SML appears to be so reasonable, intelligible, and easy, and betas can be estimated with ease, that one wonders why anyone might question the practicality of using the SML for estimating a firm's cost of equity. A risky asset's expected rate of return is the foundation for asset pricing or valuation. Some specific value for a firm's risk, it would be argued, is better than a random guess. But the apparent rationality of using CAPM as a foundation for asset pricing dissolves with careful analysis.

For example, a necessary condition for an optimal portfolio is $b\mathbf{Ax} - \mathbf{R} = \mathbf{0}$, given parameters $\mathbf{A}$, $\mathbf{R}$, and a non–zero scalar b; where, using CAPM terminology, $\mathbf{R}$ is the vector of the assets' expected excess rates of return, $\mathbf{A}$ is the matrix of covariance terms, and $\mathbf{x}$ is the vector of decision variables (i.e., weights for the assets in the optimal portfolio). If $\mathbf{A}$ is invertible, the only solution derived from this necessary condition is $\mathbf{x}^* = \mathbf{A}^{-1}\mathbf{R}/b$. The solution $\mathbf{x}^*$ depends on parameter $\mathbf{R}$, not vice versa. When the solution of $\mathbf{x}^* = \mathbf{A}^{-1}\mathbf{R}/b$ is plugged back into the necessary condition of $b\mathbf{Ax} = \mathbf{R}$, the result is a tautology. Because $\mathbf{x}^*$ is the only solution for the FOC, it must satisfy that FOC, and is thus automatically a tautology. Again, given $\mathbf{R} = b\mathbf{Ax}^* = b\mathbf{A}(\mathbf{A}^{-1}\mathbf{R})/b = \mathbf{R}$, $\mathbf{R} = \mathbf{R}$ is a tautology. Yet, if $\mathbf{x}^* \neq \mathbf{A}^{-1}\mathbf{R}/b$, then the $\mathbf{x}^*$ is not the solution for the necessary condition that $b\mathbf{Ax} = \mathbf{R}$, which then entails that $b\mathbf{Ax}^* \neq \mathbf{R}$. In other words, we can allow that $b\mathbf{Ax}^* \neq \mathbf{R}$, because $\mathbf{x}^* \neq \mathbf{A}^{-1}\mathbf{R}/b$. If $\mathbf{x}^*$ (e.g. the market index) does not result from the FOC (of $b\mathbf{Ax} = \mathbf{R}$), then the FOC (CAPM) fails.

Although $\mathbf{x}^* = \mathbf{A}^{-1}\mathbf{R}/b$ is the unique solution of the FOC $b\mathbf{Ax} - \mathbf{R} = \mathbf{0}$ ($\mathbf{x}$ is decision variable to obtain the unique solution $\mathbf{x}^*$ from the FOC, $b\mathbf{Ax}^* = \mathbf{R}$ mathematically and logically cannot be used to obtain the parameter $\mathbf{R}$. Since $b\mathbf{Ax}^*$ presupposes that $\mathbf{R}$ is known (we have the values of $\mathbf{R}$, i.e., $\mathbf{R}$ is not a dependent variable), there is no method for calculating or discovering $\mathbf{R}$ on the basis of $b\mathbf{Ax}$, which is a set of simultaneous linear functions. Similarly, we should not claim that an assets' expected rate of return depends on the market (even if it is the optimal portfolio) or its beta, as is the common practice within the CAPM framework.

This example shows that even if the solution for the FOC is correct, it still cannot solve the parameter (because of its status as a tautology). The situation is even worse should one use the wrong solution for the parameter $\mathbf{R}$. In sum, the FOC provides the means for solving the optimization problem, not for identifying the parameter of interest. The FOC is a necessary condition, not a mathematical function. We show that the optimal solution, as derived on the basis of the FOC, ensures that marginal utility of each portfolio component is zero. In fact, the expected rate of return on each asset is given by the assumption of homogeneous beliefs prior to the derivation of the CAPM. There is no rationale for further exploration of how to identify the expected rate of return on each asset.

II. THE DERIVATION OF CAPM WITHIN THE MEAN–VARIANCE FRAMEWORK

To investigate why the CAPM is dead for asset pricing, we reexamine the derivation of CAPM in the literature. As usual, assume there are n risky securities and one risk free asset in the market, the expected rate on return on i^{th} risky asset is R_i, the risk free rate of return is denoted as r, the objective function for investor is $f\,(\mu_x,\sigma_x^2)$, the expected portfolio rate of return $\mu_x = R_1x_1+R_2x_2+\ldots+R_nx_n + rx_{n+1}$ and the portfolio risk $\sigma_x^2 =$ **x'Ax** $= \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j$ for portfolio **x**, **x** is a nx1 decision variable vector with x_i as it *i*–th component, subject to the budget constraint of $(x_1 +x_2+\ldots+x_n).+x_{n+1}= 1$, x_i is the holding on asset i, i=1,2,..,n in the portfolio **x**, x_{n+1} is the holding on risk free asset, σ_{ij} is the covariance between the rate of return on i–th and j–th asset, **A** is the nxn covariance σ_{ij} matrix of rate of returns $\widetilde{R}_i$ and $\widetilde{R}_j$ on assets i and j for all i, and j.[11] We also assume homogeneity of belief about the parameters of expected rate of return on each asset and all pairwise covariances for all investors in the market. The budget constraint implies $x_{n+1} = 1-(x_1 +x_2+\ldots.+x_n) = 1-$ **e'x, e'** is the 1xn row vector with all elements being 1, the transpose of **e'** is denoted as **e**. Thus, μ_x can be rewritten as $\mu_x= r+ (R_1-r)x_1+(R_2-r)x_2+\ldots+(R_n-r)x_n$. This shows variable that the x_{n+1} can be represented by *n* independent variables of $x_1 \ldots$ to x_n, which implies that the necessary condition to maximize the objective function $f(\mu_x,\sigma_x^2)$ equals the marginal utility with respect to x_i, the investment i–th asset, is zero for all x_i, i =1, 2,…, n, i.e.,

$$\frac{\partial f(\mu_x,\sigma_x^2)}{\partial x_i} = \frac{\partial f(\mu_x,\sigma_x^2)}{\partial \mu_x}\frac{\partial \mu_x}{\partial x_i} + \frac{\partial f(\mu_x,\sigma_x^2)}{\partial \sigma_x^2}\frac{\partial \sigma_x^2}{\partial x_i}$$

$$= f_1(R_i-r) + 2f_2(\sigma_{i1}x_1 +\sigma_{i2}x_2+\ldots.+\sigma_{in}x_n) = 0, \quad \text{for all } i = 1,2, \ldots,n.$$

In the above equation, $f_1 = \frac{\partial f(\mu_x,\sigma_x^2)}{\partial \mu_x}$ and $f_2 = \frac{\partial f(\mu_x,\sigma_x^2)}{\partial \sigma_x^2}$ are, respectfully, the partial derivatives (marginal utility) of the objective function $f(\mu_x,\sigma_x^2)$ with respect to the portfolio mean μ_x and portfolio variance σ_x^2, respectively, while $\frac{\partial \mu_x}{\partial x_i} = R_i - r$, and $\frac{\partial \sigma_x^2}{\partial x_i} = 2(\sigma_{i1}x_1 + \sigma_{i2}x_2 + \cdots. +\sigma_{in}x_n)$ are the marginal portfolio mean μ_x on x_i and the marginal portfolio variance σ_x^2 on x_i, respectively, for all i = 1,2,.., n. We assume that the marginal utility of portfolio mean $f_1 > 0$, and the marginal utility of portfolio variance $f_2 < 0$ for all investors (i.e., all investors are risk–averse). For the objective function to be optimized, this necessary condition requires that the marginal utility of the portfolio mean μ_x resulting from x_i, equals the marginal utility of the portfolio variance σ_x^2 resulting from x_i for all i=1,2, … , n.[12]

This necessary condition can be stated using simultaneous linear equations as

$$\begin{aligned}
&\lambda(\sigma_{11}x_1 +\sigma_{12}x_2+\ldots.+\sigma_{1n}x_n) = R_1 - r\\
&\lambda(\sigma_{21}x_1 +\sigma_{22}x_2+\ldots.+\sigma_{2n}x_n) = R_2 - r\\
&\vdots\\
&\lambda(\sigma_{n1}x_1 +\sigma_{n2}x_2+\ldots.+\sigma_{nn}x_n) = R_n - r.
\end{aligned} \tag{1}$$

Here, $\lambda = -2f_2/f_1$ is a positive scalar; the *i*–th equation in (1) is $\lambda(\sigma_{i1}x_1 +\sigma_{i2}x_2+\ldots. +\sigma_{in}x_n) = \lambda[\mathrm{Cov}(\widetilde{R}_i, \widetilde{R}_1x_1)+\mathrm{Cov}(\widetilde{R}_i, \widetilde{R}_2x_2) +\ldots+ \mathrm{Cov}(\widetilde{R}_i,\widetilde{R}_nx_n)] = \lambda\mathrm{Cov}(\widetilde{R}_i, \widetilde{R}_1x_1+\widetilde{R}_2x_2+\ldots+\widetilde{R}_nx_n) = \lambda\mathrm{Cov}(\widetilde{R}_i, \widetilde{\mathbf{R}}'\mathbf{x}) = \lambda\mathbf{A}_i\mathbf{x} = R_i - r$, and is thus the covariance between rate of return $\widetilde{R}_i$ on asset i and the rate of return on portfolio **x**, for all i = 1, 2, …., n; $\widetilde{R}_i$ is the random variable of rate of return on asset i; σ_{ij} is the covariance between the rate of return on assets i and j; Cov(,) is the covariance operator; $\tilde{\mathbf{R}}'\mathbf{x} = \tilde{R}_1x_1 + \tilde{R}_2x_2 + \ldots + \tilde{R}_nx_n$ is the portfolio rate of return; and the 1xn row vector $\mathbf{A_i}$ stands for the *i*–th row in the nxn covariance matrix **A**.[13] For simplicity, assume **A** is invertible. The necessary condition (1) is a system of simultaneous linear equations, with n equations and n non–restrict variables **x**. It follows that equation (1) has exactly one solution. That is, according to equation (1), the optimal solution **x*** must satisfy the equation $\lambda\text{Cov}(\tilde{R}_i, \tilde{\mathbf{R}}'\mathbf{x}^*) = R_i - r$ with the same scalar λ for all i = 1,2, …, n.

In terms matrix algebra, equation (1) can be **rewritten** as (1a):

$$\lambda\mathbf{A}\mathbf{x} = \lambda\begin{bmatrix}\sigma_{11} & \cdots & \sigma_{1n}\\ \vdots & \ddots & \vdots\\ \sigma_{n1} & \cdots & \sigma_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} = \lambda\begin{bmatrix}\text{Cov}(\tilde{R}_1, \tilde{\mathbf{R}}'\mathbf{x})\\ \vdots\\ \text{Cov}(\tilde{R}_n, \tilde{\mathbf{R}}'\mathbf{x})\end{bmatrix} = \begin{bmatrix}R_1 - r\\ \vdots\\ R_n - r\end{bmatrix} = \mathbf{R} - \mathbf{r}. \qquad (1a)$$

where, $\mathbf{R} - \mathbf{r}$ is a nx1 vector of excess rates of return, and **x** is a decision variable vector of weights for investing in risky assets. This necessary condition (1) is used to solve for the optimal solution **x***, given the parameters of the expected rate of return on assets and the covariance under the constraint of $(x_1 + x_2 + \ldots + x_n) + x_{n+1} = 1$.

The unique solution for the FOC of equation (1) is

$$\mathbf{x}^* = \mathbf{A}^{-1}(\mathbf{R}-\mathbf{r})/\lambda\,. \qquad (2)$$

Equation (2) is the optimal solution for investing in risky asset selection. The portion of wealth to invest in free asset equals $1 - \mathbf{e}'\mathbf{x}^*$, given the budget constraint. Aside from indicating that the scalar λ depends on the utility function, equation (2) demonstrates that the unique solution **x*** for individual investor is determined by the covariances and expected excess rate of returns, the components of $\mathbf{A}^{-1}(\mathbf{R}-\mathbf{r})$.

To make the solution of the FOC be a portfolio merely by substitution, we replace the decision variable **x** by $\mathbf{x}/(\mathbf{e}'\mathbf{x})$ and multiply $\mathbf{e}'\mathbf{x}$ in the LHS of equation (1). Equation (1) of the FOC then can be restated as:

$$\lambda\mathbf{e}'\mathbf{x}\,(\mathbf{A}\mathbf{x})/(\mathbf{e}'\mathbf{x}) = \lambda\mathbf{e}'\mathbf{x}\begin{bmatrix}\sigma_{11} & \cdots & \sigma_{1n}\\ \vdots & \ddots & \vdots\\ \sigma_{n1} & \cdots & \sigma_{nn}\end{bmatrix}\begin{bmatrix}x_1/(\mathbf{e}'\mathbf{x})\\ \vdots\\ x_n/(\mathbf{e}'\mathbf{x})\end{bmatrix} = \lambda\mathbf{e}'\mathbf{x}\begin{bmatrix}\sigma_{11} & \cdots & \sigma_{1n}\\ \vdots & \ddots & \vdots\\ \sigma_{n1} & \cdots & \sigma_{nn}\end{bmatrix}\begin{bmatrix}\omega_1\\ \vdots\\ \omega_n\end{bmatrix}$$

$$= \lambda\mathbf{e}'\mathbf{x}\begin{bmatrix}\sigma_{11}\omega_1 & \cdots & \sigma_{1n}\omega_n\\ \vdots & \ddots & \vdots\\ \sigma_{n1}\omega_1 & \cdots & \sigma_{nn}\omega_n\end{bmatrix} = \lambda\mathbf{e}'\mathbf{x}\begin{bmatrix}\text{Cov}(\tilde{R}_1, \tilde{\mathbf{R}}'\boldsymbol{\omega})\\ \vdots\\ \text{Cov}(\tilde{R}_n, \tilde{\mathbf{R}}'\boldsymbol{\omega})\end{bmatrix} = \begin{bmatrix}R_1 - r\\ \vdots\\ R_n - r\end{bmatrix} = \mathbf{R} - \mathbf{r}, \qquad (3)$$

where, $\mathbf{e}'\mathbf{x}$ is the sum of all weight of **x** and is a 1x1 non–zero scalar, $\omega_i = x_i/(\mathbf{e}'\mathbf{x}) = x_i/(\sum_i x_i)$, the *i*–th element of 1xn vector $\mathbf{x}'/\mathbf{e}'\mathbf{x} = \boldsymbol{\omega}' = (\omega_1, \omega_2, \ldots, \omega_n)$. Equation (3) mathematically equals equation (1). The *i*–th FOC (3) is $(\lambda\mathbf{e}'\mathbf{x})\mathbf{A}_i\boldsymbol{\omega} = \lambda\mathbf{e}'\mathbf{x}(\sigma_{i1}\omega_1 + \sigma_{i2}\omega_2 + \ldots + \sigma_{in}\omega_n) = (\lambda\mathbf{e}'\mathbf{x})\text{Cov}(\tilde{R}_i, \tilde{R}_1\omega_1 + \tilde{R}_2\omega_2 + \ldots + \tilde{R}_n\omega_n) = (\lambda\mathbf{e}'\mathbf{x})\text{Cov}(\tilde{R}_i, \tilde{\mathbf{R}}'\boldsymbol{\omega}) = R_i - r$, the *i*–th element of the nx1 vector of $\mathbf{R} - \mathbf{r}$ for all i=1, 2, …, n. Similarly, the FOC of equation (3) is a set of simultaneous linear equations with n equations and n variables, which must have a unique solution if the covariance is invertible. To obtain the constant $\lambda\mathbf{e'x}$, multiply $\boldsymbol{\omega}$' into both sides of equation (3) resulting in $(\lambda\mathbf{e'x})\boldsymbol{\omega}'\mathbf{A}\boldsymbol{\omega} = (\lambda\mathbf{e'x})\sigma_\omega^2 = \boldsymbol{\omega}'(\mathbf{R} - \mathbf{r}) = R_\omega - r$, which in turn yields $\lambda\mathbf{e'x} = (R_\omega - r)/\sigma_\omega^2$, where $R_\omega - r$ and σ_ω^2 are the optimal portfolio expected excess rate of return and its variance, respectively (since the solution is unique and must be the optimal for FOC). The FOC of equation (3) can be rewritten as:

$$\frac{R\omega - r}{\sigma_\omega^2}\begin{bmatrix}\mathrm{Cov}(\widetilde{R}_1, \widetilde{\mathbf{R}}\,\acute{}\,\boldsymbol{\omega})\\ \vdots \\ \mathrm{Cov}(\widetilde{R}_n, \widetilde{\mathbf{R}}\,\acute{}\,\boldsymbol{\omega})\end{bmatrix} = (R_\omega - r)\begin{bmatrix}\frac{\mathrm{Cov}(\widetilde{R}_1, \widetilde{\mathbf{R}}\,\acute{}\,\boldsymbol{\omega})}{\sigma_\omega^2}\\ \vdots \\ \frac{\mathrm{Cov}(\widetilde{R}_n, \widetilde{\mathbf{R}}\,\acute{}\,\boldsymbol{\omega})}{\sigma_\omega^2}\end{bmatrix} = (R_\omega - r)\begin{bmatrix}\beta_{\omega 1}\\ \vdots \\ \beta_{\omega n}\end{bmatrix} = \begin{bmatrix}R_1 - r\\ \vdots \\ R_n - r\end{bmatrix} \tag{4}$$

where beta $\beta_{\omega i} = \mathrm{Cov}(\widetilde{R}_i, \widetilde{\mathbf{R}}\,\acute{}\,\boldsymbol{\omega})/\sigma_\omega^2 = [\frac{\partial\sigma_\omega^2}{\partial\boldsymbol{\omega}_i}]/(2\sigma_\omega^2)$ stands for the ratio of the covariance between the return $\widetilde{R}_i$ on *i*–th asset and the portfolio rate of return $\widetilde{\mathbf{R}}\,\acute{}\,\boldsymbol{\omega}$ over the portfolio variance σ_ω^2.[14] Hence, the beta $\beta_{\omega i}$ of *i*–th asset depends on the marginal portfolio expected excess rate of return $R_i - r$ on *i*–th asset, for all i = 1,2,3, … , n.[15] Equation (4), according to the dependency of beta, shows that the greater the parameter of expected rate of return, the greater the beta, not vice versa. In other words, to obtain a higher expected return, an investor must assume a higher risk beta to achieve it. There is no arbitrage opportunity.

To prove that CAPM is the FOC, we compare the last equality of the FOC for equation (4) with CAPM as:

$$R_i - r = \beta_{\omega i}(R_\omega - r). \qquad \text{for all i=1,2,3, … , n} \tag{5}$$

While the CAPM is

$$R_i - r = \beta_{mi}(R_m - r). \qquad \text{for all i=1,2,3, … , n} \tag{6}$$

Here, β_{mi} is the ratio of covariance between market portfolio **m** rate of return with the rate of return on *i*–th asset over the variance of market portfolio variance σ_m^2, and R_m is the expected market portfolio rate of return.

Equations (5) and (6) imply:

$$\beta_{\omega i}(R_\omega - r) = R_i - r = \beta_{mi}(R_m - r), \quad \text{for all i= 1, 2, 3, …, n}$$

or using matrix algebra terminology:

$$\mathbf{A}\boldsymbol{\omega}(R_\omega - r)/\sigma_\omega^2 = \mathbf{Am}(R_m - r)/\sigma_m^2 \;. \tag{7}$$

Given that the sum of portfolio weights is one, multiplying both sides of equation (7) by $\mathbf{e}'\mathbf{A}^{-1}$ results in $(R_\omega - r)/\sigma_\omega^2 = (R_m - r)/\sigma_m^2$. Subsequently, after multiplying

$\mathbf{A}^{-1}$ in both sides of equation (7), the market portfolio **m** must be the mean–variance efficient portfolio and equals to the unique solution $\boldsymbol{\omega}$ of the FOC (i.e., $\mathbf{m} = \boldsymbol{\omega}$).[16] This proves that CAPM is identical to the FOC with the market portfolio **m** being the unique solution of FOC. In that case, CAPM plays no role for asset pricing, because the FOC merely provides the condition for the optimal portfolio solution. Consequently, any attempt to re–state CAPM to obtain the expected cost of equity for a firm by use of the security market line (SML) is unproductive, because at that point the SML represents a portfolio constructed by a linear combination of two funds, the risk-free asset and the market portfolio (the unique mean-variance efficient portfolio), and results in a mere tautology, with no meaningful propositional content.

When we argue herein that CAPM is a tautology, we admit that the derivation of a unique optimal portfolio represents a landmark for modern finance. The extension of CAPM to the estimation of a firm's expected cost of equity by using the SML (i.e., for asset pricing) is the point at which the enterprise becomes tautologous or vacuous, simply because the market portfolio in CAPM must satisfy the FOC as well. In contrast, if the market portfolio is not the mean–variance efficient portfolio, then it cannot be the solution of FOC, and equation (6) in the CAPM derivation is violated. Equations (5), (6), and (7) prove the following proposition.

> **Proposition**: CAPM is the first order condition for optimizing the objective function of portfolio mean and variance, with the market portfolio as the unique solution for the first order condition and is not a foundation for asset pricing.

III. MARKET PORTFOLIO, MARKET EQUILIBRIUM, AND THE BETA FROM THE PERSPECTIVE OF FOC

The implication of the identity of CAPM and FOC is that other results which purportedly follow from CAPM should also be reexamined from the perspective of FOC. Those other results may not be the same as previously understood. For example, the linear relationship of the expected rate of return on an asset with its beta as in CAPM is overstated, because it is the FOC of marginal utility with respect to the portfolio decision x_i being zero $\left(\frac{\partial f(\mu_x, \sigma_x^2)}{\partial x_i} = 0\right)$ for all i = 1, 2, …, n, i.e., in the derivation of the optimal portfolio from the marginal portfolio mean and marginal portfolio variance. In particular, except for a constant scalar, beta stands for the marginal portfolio variance (or risk) which is proportional to the marginal portfolio mean when satisfying the FOC. That is, if **m** is the unique solution (i.e., the mean–variance efficient portfolio) of FOC, beta in equation (5) of FOC must be:

$$\beta_{mi} = Cov(\tilde{R}_i, \tilde{\mathbf{R}}'\mathbf{m})/\sigma_m^2 = (R_i - r)/(R_m - r) \quad \text{for all } i = 1,2,3, \dots, n \tag{8}$$

The first equality in equation (8) is beta, by definition; while the second equality results from the FOC of equation (5). Equation (8) shows that beta, which relates to the marginal portfolio variance, depends on the expected rate of return, which contradicts the fundamental assertion of CAPM. Hence, the linear equation between the expected rate of return on asset and its beta in the FOC is to solve the optimal portfolio rather than to describe the relationship between the expected rate of return and the beta. The market portfolio must be the mean–variance efficient as proven by Roll (1975) in order for CAPM to hold though it is useless for asset pricing. Since the unique mean–variance efficient portfolio is the solution of the FOC for optimizing the objective function of portfolio mean and its variance, CAPM is the FOC. It thus becomes a tautology for assets included in the FOC which satisfy the FOC, and results in a tautology when attempting to utilize CAPM for asset pricing. That is, if the asset is included in the portfolio selection and market portfolio is mean–variance efficient, then the CAPM values the asset's expected rate of return as exactly equal to the prior existent parameter of expected rate of return on the asset, because the market portfolio must satisfy the FOC.

In addition, since market equilibrium is irrelevant to the FOC under the homogeneity assumption, so is the CAPM. That is, only under the homogeneous assumption (if and only if the unique optimal portfolio solution of FOC is irrelevant to market equilibrium and is identical to the so–called market portfolio) does the FOC yield CAPM under the Kuhn–Tucker conditions. Further, portfolio variance, one of the two factors of the objective function, and the FOC contain both systematic risk and idiosyncratic risk. Therefore, the claim that idiosyncratic risk is not priced by CAPM is unsustainable, though CAPM is useless from the perspective of FOC. In fact, the distinction between these two risks from total risk is redundant, and not needed for the portfolio decision. Similarly, the extension of CAPM to the cost of debt, capital budgeting and other practices may not be justifiable either, because their basis in CAPM is unsustainable.

IV. SOCIAL SCIENCE, PAST RESEARCH AND FUTURE DIRECTIONS

Given the position we've set forth, controversies about the testability of CAPM in relation to asset pricing are irrelevant and need not continue. Since CAPM is the FOC, the FOC is always true and undeniable, because of the optimality condition in mathematics, and there is no need for further empirical testing. Testing tautologies is fruitless, and thus the testability of CAPM is not worth debating. CAPM is a tautology model by construction, given its use of the FOC to derive a parameter such as the expected rate of return. CAPM by itself, other than providing a theoretical framework for the optimal portfolio, cannot continue to be a foundation for asset pricing.

All results of empirical studies in the finance literature should be reexamined if the single market index is used when testing an asset's abnormal rate of return. In those regressions, the explanatory (independent) variable should be **independent** from dependent variable and from the error terms. Because any asset is a component of market index, the single market index model violates not only the misspecification, but also independency between the error term and the explanatory variable of market index. The market index in practice is not unique, with different sizes and components resulting in different market indexes and different results. In addition, without the rationale of CAPM, empirical results from the single market index model which use ex post market performance to explain the prior performance of individual asset rates of return are not convincing.

Recent research, including excellent literature reviews, appears to focus primarily on empirical asset pricing. Linnainmaa and Roberts (2018) provide one of these excellent reviews and comment about future research:[17]

> Two questions permeate most of the empirical asset pricing literature. The first relates to identifying a parsimonious empirical asset pricing model that provides a passable description of the cross-section of average returns; the second is about delineating between the risk-based and behavioral explanations for the many anomalies. Both lines of research can greatly benefit from the power afforded by an additional 37 years of data.

We suggest that more data do not solve the problem we present. In addition, with the strong focus on shoring up empirical issues, there appears to be very little reflection about the theoretical basis for the asset pricing promise of CAPM.

Other recent literature reviews and perhaps the large majority of recent "advances" appear to proceed without addressing the underlying theoretical problems with CAPM tracing back to Roll (1977), through Fama and French (1996), to the position we take about the tautologous nature of the basic CAPM proof. That proof legitimately provides the theoretical justification for the optimal (market) portfolio, but does not permit going beyond that portfolio to obtain expected returns for individual assets.

A response will be that finance has long gone beyond the single market model, as witnessed by on–going research, and special issues of journals devoted to CAPM, e.g., the 2012 Supplement to **Abacus**. We now have x–factor models, with the Carhart four–factor model taking on the role of the "standard" bearer within asset pricing. In addition, we have the consumption CAPM (CCAPM), the intertemporal CAPM (ICPAM), along with perhaps just about any relevant variation. But **any** legitimate asset pricing theory must offer a clear explanation about how one security's expected return escapes the problem of circularity (endogeneity), caused by the underlying tautologous nature of the CAPM proof. As Ross (1978) commented, while it may not be impossible to "test the CAPM, […] it is rather ironic that after more than a decade of study, no robust test of a supposedly testable theory exists." We second that comment, and note that it is now at least five decades of research that seemed to produce promising products. But those promises remain unfulfilled.

More serious thinking is required before we can have confidence in valuation models grounded in CAPM. The multi–factor or APT models may appear to offer hope, especially insofar as theory is occasionally offered in support of such models, but all such models are grounded in the thought that expected returns for individual assets depend upon the market's (or market mechanism) return, and that market return depends upon individual assets' expected returns. The excellent analyses in Levy (2012) and in Ferson (2019) may offer some way out.

Serious consideration of underlying theory is required. Levy offers a useful framework for future research. He provides three alternatives or courses for conceptualizing asset pricing theory (Levy, 2012, pp. 156–57, **our emphasis**):

> The **first course** relies on the positive economics approach suggested by Milton Friedman, asserting that in some models, although the underlying assumptions clearly do not hold in practice, it is still justifiable to use these models as long as investors behave "as if" these assumptions hold. The procedure to examine whether investors behave as if the assumptions are intact is done by examining the empirical fit of the data to the estimates predicted by the model under scrutiny. If there is a good fit between the theoretical estimates of the model and the observed data, then the theoretical model can be safely used despite the unrealistic assumptions made to derive it. The **second course** is a theoretical one rather than an empirical one. By this approach, one relaxes one or more of the unrealistic assumptions that underline the CAPM and derives another theoretical model that is a spinoff of the CAPM.
> […] **another course** that can be taken to handle the CAPM's unrealistic underlying assumptions is to suggest another asset-pricing model that relies on a completely different set of assumptions – for example, the Arbitrage Pricing Theory (APT) model, under which the CAPM emerges as one of the possible equilibrium solutions. In such cases, one has to evaluate the restrictions imposed by the set of assumptions corresponding to the various competing models.

While conventional wisdom would solidly affirm the wisdom of the first two alternatives offered by Levy, each falls prey to the problem of circularity we present. Only the third alternative offers the possibility of supplying the expected return for an asset in a way that may escape the tautologous nature of CAPM. Currently, the APT appears to be the strongest candidate. Yet the APT provides very little theoretical justification for choosing appropriate explanatory variables/factors. Nonetheless, using a model like APT in conjunction with Levy's first alternative (e.g., discovering the factors which provide strong explanatory power), may be the best way forward. The added benefit of an APT type of model, which may include broader macroeconomic variables, stems from the fact that such variables like GNP could capture the whole market of assets, versus a rather arbitrary selection of a market like the S&P 500. Finally, models like APT do not rely on market efficiency, thus largely escaping the many behavioral finance critiques of perfect market assumptions.

V. CONCLUSION

In order to explore why CAPM collapses in asset pricing, this paper re-examines the previous derivations of CAPM and reveals that CAPM is the necessary condition for optimizing the objective function within the mean–variance framework. The necessary condition solves the optimal solution in terms of the mean-variance efficient portfolio. It cannot be used to price assets. Taking that extra step from having an optimal portfolio to asset pricing, by using the FOC proof of CAPM, results in a tautology that undercuts the usefulness of CAPM as justifying the use of the SML.

The composition of the market portfolio used in CAPM must be identical to that of the unique mean–variance portfolio in solving the FOC for optimality, rather than simply weighted by market value or price weights, etc. In other words, if the market portfolio is the unique mean–variance efficient, which depends on the expected excess rate of return and pair–wise covariance, then the market portfolio must satisfy the FOC and the CAPM is a tautology. In contrast, if the market portfolio is not mean–variance efficient, then it will not be the solution of the FOC and the FOC is unsustainable due to the incorrect solution, and so is CAPM. Therefore, the FOC is the reason for why CAPM is useless, because the FOC cannot serve as a foundation for an asset pricing model.

ENDNOTES

[1] The necessary condition is the first order condition, and vice versa in this paper.

[2] The expected rate of return on an asset can convert into that asset's expected price, and vice versa.

[3] See p. 428 for the CAPM derivation.

[4] To maximize the utility function of the mean and variance of portfolio see eq. 3, p. 772.

[5] Merton maximizes the investor' utility function of consumption to derive his intertemporal capital asset pricing eq. 8, p. 874. The optimal portfolio is derived from eq. 19, p. 877 under the constant investment opportunity set.

[6] See the objective function Shape ratio eq.7, p. 595 and its first order condition eq. 11, p. 598

[7] Roll uses the zero-beta portfolio instead of the risk-free asset in his derivation. If the zero-beta portfolio is replaced the risk-free asset, his CAPM is the same as the CAPM derived by previous researchers.

[8] The unique solution for the necessary condition implies the market portfolio used in the CAPM must be identical to the optimal portfolio: the mean-variance efficient portfolio.

[9] For an analysis of a similar problem of endogeneity in finance, see Coles et al (2012); and as directly related to our work, see Somerville and O'Connell (2010). Others have explored or mentioned the problem of circularity in CAPM or the tautologous nature of CAPM, including Abad (2020) and Smith and Walsh (2013).

[10] The invertible matrix is a one-to-one and onto mapping.

[11] The objective function $f(\mu_x, \sigma_x^2)$ can be a utility function of portfolio rate of return μ_x and its variance σ_x^2 used by Sharpe (1964) and Mossin (1966), or a Sharp ratio used by Lintner (1965), and the rate of return subject to the constraint of portfolio variance (using the Lagrange multiplier to incorporate it) used by Roll (1977). **Bold** face as used in an equation stands for a vector or matrix.

[12] It is just like the principle that marginal revenue equals marginal cost when maximizing profit in economics.

[13] Lintner (1965, p. 599) refers to λ as the market price of dollar risk and is the same for all securities, and represents $-2f_2/f_1$; f_1 and f_2 stand for, respectfully, the partial derivatives of portfolio mean and variance or the marginal utility of portfolio return and its risk as used by Mossin (1966, p. 772). λ is the Lagrange multiplier in mathematical programming and is the ratio of the portfolio expected excess rate of return over portfolio risk to maximize the Sharpe Ratio of the objective function.

[14] Or by Kuhn-Tucker conditions, which states the $x_i(\frac{\partial f(\mu_x,\sigma_x^2)}{\partial x_i}) = x_i[\frac{\partial f(\mu_x,\sigma_x^2)}{\partial \mu_x}\frac{\partial \mu_x}{\partial x_i} + \frac{\partial f(\mu_x,\sigma_x^2)}{\partial \sigma_x^2}\frac{\partial \sigma_x^2}{\partial x_i}] = 0$ for all i = 1,2,…,n.

[15] The solution of the FOC $\boldsymbol{\omega} = \mathbf{A}^{-1}(\mathbf{R}-\mathbf{r})/[\mathbf{e}'\mathbf{A}^{-1}(\mathbf{R}-\mathbf{r})]$ is the unique mean-variance efficient portfolio.

[16] See Merton (1973, p. 878, eq. 20), Roll (1977, p. 166, eq. A7 and p. 160 eq. A18) when zero beta portfolio is replaced by risk free asset, Sharpe (1964, p. 434, Fig. 5), and Lintner (1965, p. 600, eq. 17–19) where he states: 'all investor's portfolio value (8), assume homogeneous belief for all, and set equal to the market portfolio value in order to get the CAPM (17a or 19 for pricing) in equilibrium.'

[17] Linnainmaa and Roberts (2018), p. 2641.

REFERENCES

Abad, P. (2020), 'Arbitrage Opportunity, Impossible Frontier, and Logical Circularity in CAPM Equation,' https://ssrn.com/abstract=3542055.

Black, F., M.C. Jensen, and M. Scholes (1972), 'The Capital Asset Pricing Model: Some Empirical Tests', in *Studies in the Theory of Capital Markets*, Ed. Michael C. Jensen, New York: Praeger.

Coles, J.L., M.L. Lemmon and J.F. Meschke (2012), 'Structural Models and Endogeneity in Corporate Finance: The Link Between Managerial Ownership and Corporate Performance," *Journal of Financial Economics*, Vol. 103, pp. 149–168.

Fama, E.F., and K.R. French (1996), 'The CAPM is Wanted, Dead or Live', *Journal of Finance*, Vol. 51, pp. 1947–1958.

Fama, E.F., and J.D. MacBeth (1973), 'Risk, Return, and Equilibrium: Empirical Tests', *Journal of Political Economy* 81, Vol. 3, pp. 607–636.

Ferson, W. (2019), *Empirical Asset Pricing: Models and Methods*, The MIT Press, Cambridge, Massachusetts.

Harvey, C.R., Y. Liu and H. Zhu (2016), '… and the Cross–Section of Expected Returns,' *The Review of Financial Studies*, Vol. 29, Issue 1, pp. 5–68.

Hur, J. and R. Kumar (2007), 'Beta May Not Be Dead After All: A New Framework for Measuring and Correcting the Bias in Cross–Sectional Test of the CAPM', *Working Paper*.

Jagannathan, R. and Z. Wang (1994), 'The CAPM is Alive and Well', SSRN: http://ssrn.com/abstract=5444.

Kothari, S.P., J.A. Shanken, and R. Sloan (1998), 'The CAPM: "Reports of My Death Have Been Greatly Exaggerated", SSRN: http://ssrn.com/abstract=6975.

Lai, T–Y, and M.H. Stohs (2015), 'Yes, CAPM Is Dead' *International Journal of Business*, Vol. 20, No. 2, pp. 144–164.

Levy, H. (2010), 'The CAPM is Alive and Well: A Review and Synthesis,' *European Financial Management*, Vol. 16, No. 1, 43–71.

Levy, H. (2012), *The Capital Asset Pricing Model in the 21st Century; Analytical, Empirical, and Behavioral Perspectives*, Cambridge University Press, New York.

Linnainmaa J.T. and M.R. Roberts (2018), 'The History of the Cross–Sections of Stock Returns," *The Review of Financial Studies*, Vol. 31, No. 7, pp. 2606–2649.

Lintner J. (1965), 'Security Prices, Risk and Maximal Gains from Diversification,' *Journal of Finance,* Vol. 20, December, pp. 587–615.
Merton, R. (1973), 'An Intertemporal Capital Asset Pricing Model', *Econometrica,* Vol. 41, No. 5, pp. 867–887.
Mossin J. (1966), 'Equilibrium in a Capital Asset Market,' *Econometrica,* Vol. 34, pp. 768–783.
Roll, R. (1977), 'A Critique of the Asset Pricing Theory's Tests–Part 1: On Past and Potential Testability of the Theory,' *Journal of Financial Economics,* Vol. 4, pp. 129–176.
Ross, S. (1978), 'The Current Status of the Capital Asset Pricing Model (CAPM)' *Journal of Finance*, Vol. 33, No. 3, pp. 885–901.
Sharpe, W. F. (1964), 'Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk,' *Journal of Finance,* Vol. 19, pp. 425–442.
Smith, T. and K. Walsh (2013), 'Why the CAPM is Half – Right and Everything Else is Wrong," *Abacus,* Vol. 49, Supplement, pp. 73–78.
Somerville, R.A. and P.G.J. O'Connell (2002), 'On the Endogeneity of the Mean – Variance Efficient Frontier,' *The Journal of Economic Education*, Vol. 33, Issue 4, pp. 357–366.