A MODEL FOR CORPORATE BONDS AND THE PRICING OF INTEREST RATE SWAPS WITH CREDIT RISK

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This paper proposes a simple approach to value corporate debts and applies it to price interest rate swaps with default risk. Our approach is motivated by the observation that default premiums are closely related to the state of economy so that we may use a market index to reflect the default behavior of a company. We use the instantaneous interest rate and the market index as state variables to derive the valuation equation for an interest rate contingent claim. Different default behaviors are characterized by different default boundaries. The main advantage of our approach is that we can deal with ease an interest rate contingent claim involving two counterparties such as a swap. Since it is straightforward to use the observed corporate credit spreads to characterize the default boundaries, this model can serve as a practical tool for the purpose of various credit risk management.

I. INTRODUCTION

Corporate bonds are priced to realize higher yields than comparable Treasury issues because of the possibility of default. Traditional approach to valuing the risky debts typically assumes two stochastic processes: one on the stochastic behavior of the interest rates, the other the random evolution of the firm's value. A two factor arbitrage pricing method is then applied to derive the valuation partial differential equation where the credit risk is incorporated into various terminal and boundary conditions depending on the treatment of events which trigger the default. For instances, Black and Cox [5], Brennan and Schwartz [6], Jones, Mason and Rosenfeld [16], and Merton [18] define that bankruptcy occurs whenever the firm's value decreases to the level of the debts. The empirical study along this line is, however, not very supportive

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(Jones, Mason and Rosenfeld [16]). Kim, Ramaswamy, and Sundaresan [20], on the other hand, recognize that the cash flow problem is actually the source of financial distress and therefore bankruptcy is precipitated if the firm's cashflows are unable to cover its interest obligations. All these models use the process of the firm's value as the state variable to capture the behavior of corporate default, and the estimation of the firm's value process is an intricate issue.

In valuing the corporate floating rate instruments, Ramaswamy and Sundaresan [20] also consider an alternative approach. They treat instantaneous default premium as a state variable and assume its process to be mean-reverting to account for the observed behavior of the default premiums for the money market instruments. Nevertheless, their model does not endogenize the bankruptcy event since the default premium process is exogenously given.

To incorporate the features of default risk into an arbitrage pricing model, we propose a new approach in which a proxy of economic condition, rather than the value of the firm, is used as the state variable which generates default. Since there are abundant empirical evidences (Fama [9], Fons [11], Jaffee [15] etc.) that the default premiums are closely related to the stage of business cycle, it is reasonable to argue that a process proxying the state of the economy may capture much of the behavior of the default. The model constructed in this paper differs from the standard contingent claims model for the risky debts in the way in which we specify the occurrence and implications of bankruptcy. It turns out that our approach for the risky debts (i) enables us to utilize the observed *risk structure* of interest rates to price default-risky interest rate contingent claims; (ii) significantly facilitates the valuation of such claims of which the credit risk is *bilateral* as interest rate swaps; (iii) provides a relative pricing model of risky derivative securities consistent to the credit analysis practice observed in the industry.

The paper is organized as follows. In Section 2, we propose an arbitrage model to value the corporate bonds and compare it with the traditional approach. Particularly, we provide insight for pricing the default-risky contingent claims relative to the observed risk structure in the arbitrage framework. In Section 3, we recapitulate the fast growing interest rate swap market and apply our corporate bond valuation model to price the interest rate swap with credit risk. In section 4, we take S&P 500 as market proxy and numerically solve the resulting partial differential equation. The equilibrium swap rates under various scenarios are reported and contrasted with other studies in the literature. Section 5 is an extension of the model to the pricing of callable swaps and swaptions. Section 6 concludes the paper.

II. A MODEL FOR VALUING RISKY DEBTS

An Arbitrage Pricing Framework

We now describe the risky debt valuation model which will be used to price the default-risky interest rate swaps in the next section. The following assumptions are employed:

(A.1) Trading takes place continuously in frictionless markets; there are no taxes, transaction costs or informational asymmetries.

(A.2) The term structure is fully specified by the instantaneously riskless rate $r(t)$. Its dynamics are given by

$$dr = \kappa(\mu_r - r)dt + \sigma_r r^{1/2}dz \tag{1}$$

where $\kappa,\ \mu_r,\ \sigma_r > 0$ and $\{z_r(t),\ t \geq 0\}$ is a standard Wiener process.

(A.3) The process to proxy the economic condition, market portfolio for instance, is assumed to follow a CEV (Constant Elasticity of Variance) diffusion

$$dx = \mu_x x dt + \sigma_x x^{\gamma/2} dz_x \tag{2}$$

where $\mu_x,\ \sigma_x > 0$, $0 < \gamma < 2$, and $\{z_x(t),\ t \geq 0\}$ is a standard Wiener process which may be correlated with the process $\{z_r(t),\ t \geq 0\}$. When $\gamma = 2$, the process is lognormal.

(A.4) The bankruptcy of the firm is triggered by the event that the level of $x(t)$ decreases to certain value. Explicitly, let $k(t)$ are critical values of $x(t)$ that the firm encounters financial distress, then $x(t) \leq k(t)$ implies that the firm fails to disburse its interest or principal payments and bankrupts.

The Valuation Equation

We have seen that the underlying state variables in the model are the interest rate $r(t)$ and the proxy of economic condition $x(t)$. The bankruptcy events are precipitated by a default boundary $k(t)$. Therefore, we can represent the value of a corporate bond as $P(r,x,t;k,c)$ where t is time index and c is the coupon rate. By applying the arbitrage argument as shown in **Appendix**, the value of the corporate bond must satisfy the following partial differential equation:

$$\tfrac{1}{2}\sigma_r^2 r P_{rr} + \rho\sigma_r\sigma_x r^{1/2} x^{\gamma/2} P_{rx} + \tfrac{1}{2}\sigma_x^2 x^{\gamma} P_{xx} + [\kappa(\mu_r - r) - \lambda r]P_r + rx P_x + P_t - rP = 0, \tag{3}$$

where λr represents factor risk premium.

The value of the corporate bond is also required to satisfy boundary conditions as follows:

(B.1) The payoffs to the bondholders upon bankruptcy is a fraction, $\delta x/k$, of the value of comparable Treasury bond $B(r,t;c)$.

$$P(r,x,t;k,c) = (\delta x/k)B(r,t;c) \quad \text{if } x(t) \le k(t),$$

where $0 \le \delta \le 1$ is a scale parameter to be estimated and $B(r,t;c)$ is obtained directly from Assumption (A.1) and Cox, Ingersoll, and Ross [8].

(B.2) At maturity T, the corporate bondholders receive $\delta x/k$ of face value (assuming \$ 1) if the firm is insolvent, or the face value otherwise.

$$P(r,x,T;k,c) = \begin{cases} \delta x/k & \text{if } x(T) \le k(T) \\ 1 & \text{if } x(T) > k(T) \end{cases}$$

(B.3) As x approaches to infinity, the payoff of risky bond approaches the value of an otherwise identical Treasury bond.

$$\lim_{x \to \infty} P(r,x,t;k,c) = B(r,t;c)$$

The Determination of Default Boundary

So far we have not discussed how the default boundary $k(t)$ is determined except that it is firm-specific. If such boundary were given exogenously, the model could just be implemented in a straightforward way as described above. This is however hardly the case. The firm's default boundary simply does not reveal to us because it describes the firm's possible default behavior at each instant of time whereas the firm goes to bankruptcy only at certain point of time if bankruptcy did occur. Therefore, to close the model, we must derive the default boundary endogenously by utilizing some observable default information of the target firm.

We begin with this task by arguing that the corporate *default probability* is observable in the sense it can be measured by the historical default experience in the category of the firms with similar quality rating as done in Altman [1]. We then associate the default boundary with the default probability through the conditional density of the market proxy process as follows:

Definition 1 *Given assumptions (A.3) and current level of market proxy x(t), if the default probability of the firm at time t is denoted by p(t), then the firm's default boundary, k(t), is defined by the following relationship:*

$$p(t) = Prob\{x(t+\tau) \leq k(t+\tau)|x(t)\} \quad \forall \tau \geq 0 \qquad (4)$$

The definition says that, given today's information on market proxy, the conditional probabilities that the market proxy may drop below the default boundary at each instant are exactly the firm's current default probability. Therefore, once the firm's default probability based on today's information is determined, the default boundary which triggers bankruptcy is fixed and the price of corporate bond is subsequently determined by the partial differential equation and relevant boundary conditions described above.

Pricing Default Risk of the Interest Rate Contingent Claims

The model has the advantage over the traditional approach for corporate bonds in that it can be extended to price the default-risky contingent claims in a more practical manner. Analogous to the approach pioneered by Ho and Lee [13] who valued the default-free interest rate contingent claims relative to the initial term structure, our model determines the default premiums for interest rate contingent claims relative to the *current risk structure*. The crux is that we first recover a firm's default boundary through our bond valuation model by utilizing its credit spread observed in the market, then price the default risk of contingent claims according to the derived boundary. Therefore, when we value the default-risky contingent claims, the model's bond prices of the contract firms are guaranteed to match those actually observed.

The traditional approach requires the input of the complete dynamics of the firm's value to endogenize the default events. Our approach models the default behavior of a firm through the state of the economy and an implied default probability of individual firm. Therefore, we allot some burden of acquiring default information on the market proxy which is more publicly observable. Since we require less knowledge (default probability rather than value process) about the firm, we have more freedom to match the *current risk structure* of interest rates.

From the perspective that the default risk of the interest rate contingent claims is priced according to the current risk structure, our approach is analogous to Ho and Lee's [13] work. Nevertheless, the scope of our paper is much wider. They priced the interest rate risk of the interest rate

derivatives taking today's term structure as given, there is no warranty that the initial term structure itself is arbitrage free. In contrast, Our approach uses today's risk structure to recover the default probability (and boundary) through a corporate bond valuation model which is arbitrage free, then we proceed to price the default risk of derivatives relative to this boundary. Therefore, the default boundary served as a benchmark is assured to be arbitrage free under our framework. In essence, the reason we would like to retrieve the default boundary from the observed risk structure is not because we do not have a model for the risk structure, it is because we do not have a reliable estimate for firm's default probability.

We now turn to the valuation of a default-risky interest rate swap contract, how the relative pricing described above can be implemented will be further elaborated there.

III. VALUATION OF INTEREST RATE SWAPS WITH CREDIT RISK

Interest Rate Swap Market

In the last two decades there has been a dramatic increase in the number of derivative securities, such as options, futures, and financial swaps, etc. Swaps are commonly portrayed as one of the latest financing innovations because they were not publicly introduced until as recently as 1981. A swap contract obligates two parties to exchange some specified cash flows at specified intervals.

The most common form of swaps is the interest rate swap in which two counterparties agree to exchange a sequence of cash flows representing fixed- and floating-rate (or different floating-rate) interest payments on an agreed-upon principal amount. While interest rate based derivative instruments such as interest rate options and futures experienced a vastly increasing demand during the past decade of volatile interest rates, the interest rate swap market has also grown explosively from virtual non-existence in 1981 to approximately $350 billion in 1985, and $889.5 billion in 1988.

The Credit Risk of the Interest Rate Swaps

The rapid growth and the sheer volume of the interest rate swap contracts, combined with the volatility in financial markets, brought the question of credit risk of such instruments into sharp focus. It also caused regulators and

monetary authorities to consider the inclusion of swaps in their assessment of bank capital adequacy. Therefore, an effort of valuation of swaps with credit risk appears to be especially timely.

Credit risk exists in swap contracts because of the possibility that a counterparty may default on its obligation after an unfavorable movement in interest rate. In this section, we will apply the corporate bond valuation approach proposed in the previous section to evaluate the credit risk associated with interest rate swaps, and to use this framework to address pricing and capital adequacy issues associated with these contracts.

Swaps are different from bank loans in that they do not involve the advancement of principal to a counterparty. Thus, it is never appropriate to use the notional amount of a swap as a measure of credit risk. A more important feature of swap credit risk is its bilateral nature. Interest rate swaps can lead to loss to a participant only if two events occur together: *First*, the counterparty to the swap must fail to perform according to the terms of the swap contract. This is often referred to as *bankruptcy risk. Second*, interest rates must move adversely to those agreed in the original contract, implying a cost in replacing the cash flows from the original swap contract. In other words, the swap contract must be out of the money at the time of default. This is referred to as *mark-to-market risk* which results from the fluctuation in the replacement cost of the swap in response to changes in interest rates.

There have been many earlier works devoted to swap pricing with credit risk. Belton [3], Ferron and Handjinicolaou [10] estimate the maximum probable loss on swaps, but do not attempt to value the default risk using an equilibrium model. Whittaker [27] value the credit exposure of interest rate swaps using no-arbitrage argument, but does not endogenize the event triggering the swap default. His results is, therefore, measures of the value of swap default assuming that the probability of the event triggering default is independent of the size of the default. Cooper and Mello [7] develop a partial equilibrium model for swap default that is consistent with the traditional approach for valuing the risky debts, but they treat only one counterparty of the swap contract to be risky. Hull [14], also assuming only one counterparty to be risky, treats default as an option held by the risky counterparty and values a default-risky swap by subtracting this default option from the value of a default-free swap. He however makes strong assumption that the state variables affecting the probability of bankruptcy have zero market price of risk and are independent of the interest rate uncertainty. Sundaresan [23] values the default risk of swap by introducing an instantaneous default premium process into his default-free swap pricing model.This is an application of the approach originally developed in Ramaswamy and Sundaresan [20]. Like Whittaker's [28] approach, the

model does not endogenize the default event since the default premium is exogenously given. In any event, all previous efforts tend to ignore the bilateral nature of swap credit risk, our model, alternatively, is an improvement toward this direction.

Pricing Default-Risky Interest Rate Swaps

For ease of exposition, the interest rate swap contract considered in this section is the simplest type of swap contract which involves a highly rated firm with access to the fixed rate bond market and a lower rated firm in need of fixed rate funds. The low rated firm, also referred to as the floating-rate receiver, would borrow short term in the floating rate market and make fixed rate swap payments to the highly rated firm that are a fixed percentage of a notional principal amount. The highly rated firm, referred to as the fixed-rate receiver, agrees to send its counterparty payments that are a floating percentage (calculated by a floating-rate index) of the notional principal amount. No underlying principal is exchanged in the swap transaction and only net payments are made at each semiannual settlement date.

Suppose that the highly rated firm promises to pay a floating rate $r(t)$ to the lower rated firm in exchange for a fixed rate $c(t) \equiv c$. The value of this contract at time t will depend on the floating rate $r(t)$, the fixed rate c, and the economic condition process $x(t)$ which, as mentioned before, is argued to capture the behavior of default of both firms. Notationally, the value of this swap contract can be written as $S_t = S(r_t, x_t, t; c)$. Since both counterparties of the swap contracts are subject to credit risk, using our model in the previous section, the bankruptcy of each firm will be triggered by the events that the levels of $x(t)$ decreases to their corresponding default boundaries, $k_h(t)$ for the higher rated firm and $k_l(t)$ for the lower rated firm. Formally, we have the following definition:

Definition 2 *The default of the lower rated firm (floating rate receiver) is defined by the event $x(t) \leq k_l(t)$, while that of the highly rated firm (fixed rate receiver) is defined by the event $x(t) \leq k_h(t)$. Assuming $k_h(t) < k_l(t)$.*

As indicated earlier, the swap holder will not default on the contract as long as it is an asset (ie. has positive value) for the holder. Likewise, the holder of a negative value swap may not be detached from the obligation even the counterparty faces the financial distress. Therefore, the swap value *with respect to the higher rated firm, $S(r, x, t; c)$*, must satisfy the following terminal condition:

$$c - r(T) \quad \quad if\, x(T) > k_l(T)$$

$$S(r,x,T;c) = \int \begin{cases} c - r(T) & \text{if } c-r(T)\leq 0,\ k_h(T)<x(T)\leq k_l(T) \\ 0 & \text{if } c-r(T)>0,\ k_h(T)<x(T)\leq k_l(T) \\ 0 & \text{if } x(T)\leq k_h(T) \end{cases} \tag{5}$$

where T is the maturity date of the swap contract. This condition says that at maturity, the value of the swap contract in term of the highly rated firm, $S(r,x,T;c)$, depends not only on the magnitude of the prevailing floating rate, $r(T)$, and fixed rate, c, but on the bankruptcy conditions of two firms. When both firms are solvent ($x(T)>k_l(T)$), the swap value is either positive or negative depending on the rate movement as in the case of no credit risk. When both firms are insolvent ($x(T)\leq k_h(T)$), the swap value is obviously zero. When the highly rated firm is solvent but the lower rated firm is insolvent ($k_h(T)<x(T)\leq k_l(T)$), the swap value to the highly rated firm can only be negative or zero. Although the lower rated firm has bankrupted, the in-the-money swap is generally honored by the highly rated firm or it can be sold in the secondary market whereas the out-of-the-money swap is simply defaulted due to the insolvency of lower rated firm.

By the same argument, the swap value at any time before maturity must satisfy the following boundary conditions:

$$S(r,x,t;c) = \int \begin{cases} c-r(t)+S(r,x,t_+;c) & \text{if } x(t)>k_l(t) \\ c-r(t)+S(r,x,t_+;c) & \text{if } c-r(t)+S(r,x,t_+;c)\leq 0,\ \text{and} \\ & \quad k_h(t)<x(t)\leq k_l(t) \\ 0 & \text{if } c-r(t)+S(r,x,t_+;c)>0,\ \text{and} \\ & \quad k_h(t)<x(t)\leq k_l(t) \\ 0 & \text{if } x(t)\leq k_h(t) \end{cases} \tag{6}$$

The modeling of terminal and boundary conditions as above distinguishes our model from the earlier research in the literature: we have modelled the fact that a swap counterparty defaults on its obligation if bankruptcy is declared *and* the swap value is negative. An essential feature of the interest rate swap is that it involves two counterparties and each of them has the choice to default the contract, this is fundamentally different from the default of the risky debts in which only the corporate issuers will default on the contracts. A straightforward extension of traditional corporate bond approach to value the swap credit risk would be to introduce another stochastic process to represent the value of the second firm, that is, there are two processes describing the values of the two counterparties respectively. Unfortunately, the introduction of extra state variable would render the valuation problem quite intractable. Our approach, instead of expanding the

dimension of state variables, includes the bilateral nature of swap credit risk into the terminal and boundary conditions by arguing that the event which precipitates bankruptcy is closely related to the situation of the economy as a whole.

The Valuation Equation for Default Risky Swaps

The underlying state variables in our model are the interest rate r and the proxy of economic condition x. Given the assumptions (A.2), (A.3) and fixed rate c, the value of the interest rate swap S must satisfy the following partial differential equation:

$$\tfrac{1}{2}\sigma_r^2 r S_{rr} + \rho\sigma_r\sigma_x r^{\frac{1}{2}} x^{\gamma/2} S_{rx} + \tfrac{1}{2}\sigma_x^2 x^\gamma S_{xx} + [\kappa(\mu_r - r) - \lambda r] S_r + rx S_x + S_t - rS = 0, \quad (7)$$

where λr represents factor risk premium.

For a given c, the equation is then solved subject to the terminal and boundary conditions described above. Through a search procedure, the equilibrium swap rate is a c^* such that

$$S(r,x,0;c^*) = 0$$

Default Boundaries for Risky Swaps

One of the key features of our framework is to model the relative riskiness of the two participants of the swap contract through appropriate calibration of bankruptcy boundaries $k_h(t)$ and $k_l(t)$. If we are able to figure out the default probabilities p_h and p_l for higher rated and lower rated firms respectively, then we can simply use the definition in equation (2) to obtain the boundaries and solve the valuation equation accordingly. Nevertheless, people may argue that these probabilities are difficult to measure and the reliability is also a problem. The sample space of default incidence is just too small to make credible inferences. Fortunately, in our framework, we can very easily recover the implied default probabilities and default boundaries by looking at the yield spreads of corporate bonds over their comparable Treasury issues. Because such spreads are typically used by practitioners and researchers as measures of relative riskiness among firms, the derived default boundaries, and hence the value of the swap, should significantly reflect the credit risk conjectured by the market. We now briefly discuss how the derivation of default boundaries from credit yield spreads of corporate bonds can be proceeded. Given the parameters of the interest rate process and market proxy process as in (A.2) and (A.3), we first use the current values of short

rate, *r(t)*, and interest rate process to obtain the default-free par bond yield, denoted by ù , prevailing on the market. Then, adding the observable credit yield spreads of higher and lower rated firms, denoted by ε_h and ε_l respectively, on the Treasury par bond yield we arrive at the par bond yields for debt issues of those two firms. For higher rated firm, We set the coupon payment equal to its par bond yield (ù+ε_h) and then perturb the implied default probability (which is equivalent to perturbing the default boundary) until this bond is valued at par. This procedure gives the default boundary for higher rated firm. By the same token, the default boundary for lower rated firm can also be derived. Therefore, when we price default-risky swaps under current framework, the model's bond prices of the both counterparty firms are guaranteed to be those actually observed.

The reason that the traditional approach of corporate bond valuation does not work, or at least very difficult to proceed, for our purpose is that we have to recover a stochastic process (firm's value process) in order to match the observed credit spreads. Even for a lognormal process, searching over two parameters (drift and diffusion coefficients) may cost huge amount of computations. Our approach only searches for a probability which is, by nature, confined between 0 and 1. Our experience in solving the swap rates as reported in the next section is that the convergence can normally be achieved in several iterations.

Intuitively, our approach performs very differently from the traditional approach in that the traditional approach is more dependent on the individual firm's information. Instead of requiring full information of the dynamics of the value of the firm, we allot some burden of acquiring default information on the market proxy which is more publicly observable. Because we require less default knowledge (default probability rather than value process) about the individual firm, we are able to back out such knowledge efficiently from the observed credit spreads. It may be arguable that the new approach essentially does not improve the valuation of corporate bonds in terms of the information content, it is obvious that our approach is more applicable for pricing default-risky contingent claims in a practical way.

IV. NUMERICAL RESULTS FOR THE EQUILIBRIUM SWAP RATES

Data and Parameter Estimates

Our model for pricing risky swaps requires three sets of input data: estimates for parameters of interest rate process, estimates for parameters of market

proxy process, and the credit spreads of corporate bonds for higher and lower rated firms. We obtained the estimates of parameters of CIR interest rate process as in (A.2) from Pearson and Sun [19], who estimated a two factor CIR model by maximum likelihood based on the conditional density of unobservable state variables through bond price formulae. The configuration of estimates is as follows:

$$\mu_r=0.07936; \; \sigma_r=0.11425; \; \kappa=0.29368; \; \lambda=0; \; \delta=0.$$

The market proxy in current study was chosen to be S&P500 for its popularity. The data set we used for this index was the monthly quotations from January 1970 to June 1991, 258 observations in total. We estimated the parameters of the CEV process and found that the speed of adjustment parameter γ was very close to 2. We then decide to make S&P500 process lognormal in current study. The estimation of drift and diffusion coefficients of a lognormal process is straightforward, and we came up with estimates:

$$\mu_x=0.08196; \; \sigma_x=0.16217$$

Because swaps are usually transacted among A or better rated firms, we select AAA quality rating to correspond to our higher rated firm and A quality rating to correspond to lower rated firm. To determine the credit spreads of AAA and A rated firms over the comparable Treasury, we collected the daily quotations reported by Lehman Brothers International of such spreads for an AAA firm (Exxon) and an A firm (Ford). The daily quotations covered the period 1/8/91 to 5/21/91. The means of those spreads are 38 basis points for Exxon and 81 basis points for Ford for the 2 year bonds. For the bonds with maturity of 10 years, the spreads widened to 60 basis points for Exxon and 111 basis points for Ford. In the current version, we will report the empirical results only for 2 year and 10 year swaps. The results for intermediate maturity will be presented in the next version.

Finally, we assume that today's short rate was 6% and the current S&P index is 370. The magnitudes of swap default spreads and their behavior given above parameter structure are summarized in the following subsection.

Numerical Results

Given today's short rate 6%, it can be derived from the interest rate process that the implied par bond yield for the Treasury should be 6.5368%. Adding the credit spreads we have calculated earlier on the Treasury par bond yield,

we obtained the implied par bond yields for AAA and A rated firms with 2 and 10 year maturity respectively. The spreads and implied par bond yields are exhibited in Table 1.
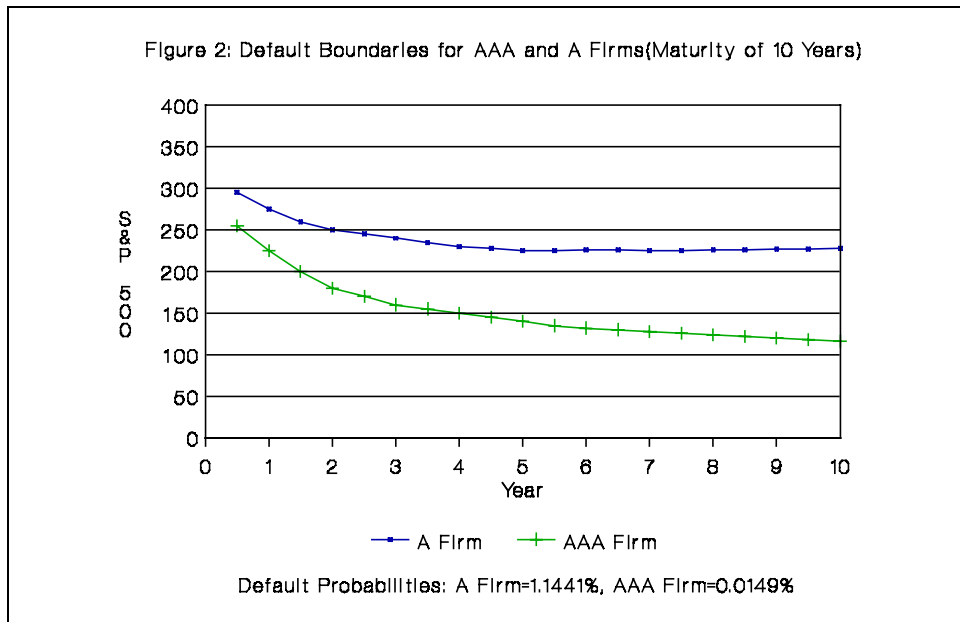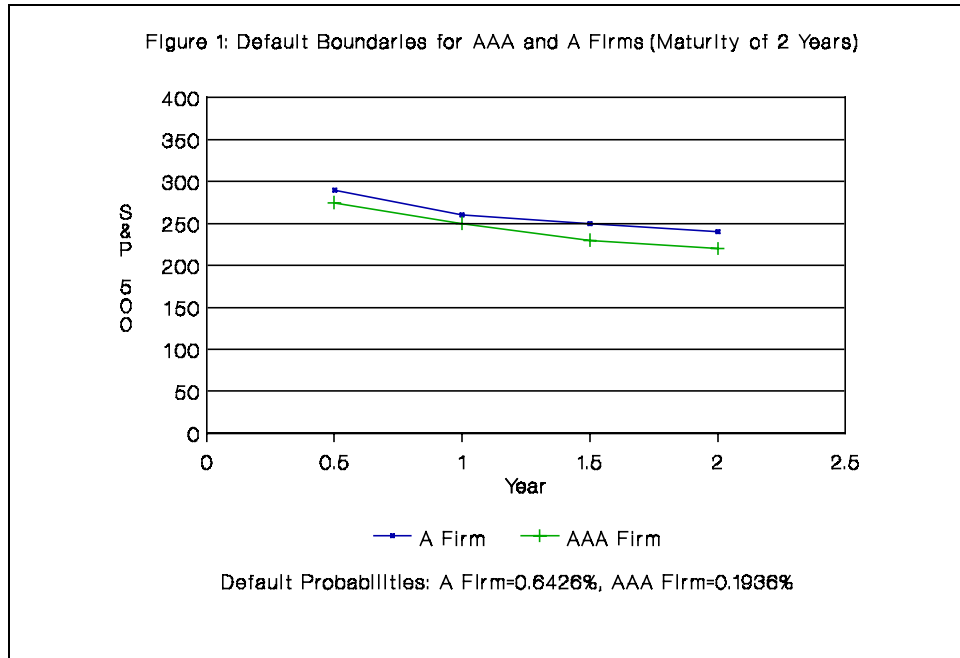
Table 1: Par Bond Yield and Default Probability[1]

Maturity of 2 years			
Credit Rating	Treasure	AAA Firm	A Firm
Credit Spread	0	38 bp	81 bp
Par Bond Yield	6.5368%	6.9168%	7.7268%
Default Probability	0	0.1936%	0.6426%

Maturity of 10 years			
Credit Rating	Treasure	AAA Firm	A firm
Credit Spread	0	60 bp	111 bp
Par Bond Yield	6.5368%	7.1368%	8.2468%
Default Probability	0	0.0149%	1.1441%

Take the 2-year par bond yield of the AAA rated firm, 6.9168%, as the coupon rate of a corporate bond issued by AAA firm. Using our corporate bond model, we were able to find out the magnitude of the implied default probability which made the bond priced at par.The default boundary for 2-year bond of AAA firm was simultaneously derived. Default probabilities and boundaries for other bonds were similarly derived. Table 1 lists the resulting default probabilities. Figure 1 and Figure 2 provide the graphs of default boundaries for 2 and 10 year corporate bonds respectively.

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[1]. The parameters for the model are as follows: $m_r = 0.07936$, $s_r = 0.11425$, $k = 0.29368$, $l = 0$, $d = 0$, $m_x = 0.08196$, $s_x = 0.16217$, and $r = -0.2$. Current short rate is 6% and current market index is 370.

Figure 1: Default Boundaries for AAA and A Firms (Maturity of 2 Years)

Default Probabilities: A Firm=0.6426%, AAA Firm=0.1936%

Figure 2: Default Boundaries for AAA and A Firms(Maturity of 10 Years)

Default Probabilities: A Firm=1.1441%, AAA Firm=0.0149%

The default probabilities vary from 0.1936% to 0.6426% for 2 year bonds and 0.0149% to 1.1441% for 10 year bonds. These numbers are contrasted to the historical corporate bond default rates reported by Altman (1985). He reported the average default rates of 0.160% for all bonds and 2.240% for bonds rated BBB and below, based on the data between 1970 and 1984. He, however, did not break historical default probabilities into sectors of different maturities. Overall, the default probabilities generated by our model are quite consistent with the historical experience. This fact makes us considerably confident on our swap pricing results.

The equilibrium swap rates and credit spreads under various scenarios are reported on Table 2 and Table 3. The upper panel of Table 2 gives the 2-year swap rates an AAA firm would require if it promised to make floating rate payments to an A rated firm (line 1) or to Treasury-type (riskless) firm (line 2). The difference between line 1 and 2, or 0.81 basis points, is interpreted as the default spread required by an AAA firm if it trades with an A firm. In other words, the default risk of the A rated firm was priced by 0.81 basis points by the AAA firm. The middle panel are interpreted analogously.

The lower panel of Table 2 warrants some explanation. 6.5196% stands for the swap rates when both counterparties are default-free. Since the swap rate in line 4, 6.5025%, is the rate an A firm would require if it traded with Treasury, the difference between this two swap rates (i.e. line 5 minus line 4) must be the default risk the A rated firm bore with. Notice that this difference amounts to 1.71 basis points which is much larger than 0.81 basis points seen in the upper panel. The reason that the AAA firm measured the default risk of the A rated firm with lower premium is that the *AAA firm was also default-risky*.

Table 2: Two Year Swap Rates and Default Spreads[2]

AAA Firm Receives Fixed Rate				
Line	Counterparty	Swap Rate	Spread	
(1)	A Firm	6.5236%[3]	0.81 bp	(1) - (2)

(2)	Treasury	6.5155%		

A Firm Receives Fixed Rate				
Line	Counterparty	Swap Rate	Spread	
(3)	AAA Firm	6.5052%	0.27 bp	(3) - (4)
(4)	Treasury	6.5025%		

Treasury Receives Fixed Rate				
Line	Counterparty	Swap Rate	Spread	
(5)	Treasury	6.5196%	0.41 bp	(5) - (2)
(6)			1.71 bp	(5) - (4)

Table 3: Ten Year Swap Rates and Default Spreads[4]

AAA Firm Receives Fixed Rate				
Line	Counterparty	Swap Rate	Spread	
(1)	A Firm	7.0483%[5]	2.84 bp	(1) - (2)
(2)	Treasury	7.0199%		

A Firm Receives Fixed Rate

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[3]. The rate here represents the fixed rate the AAA firm must receive to compensate its floating payments to an A rated firm. Other swap rates are interpreted similarly.

[4]. The parameters for the swaps are as follows: $m_r = 0.07936$, $s_r = 0.11425$, $k = 0.29368$, $l = 0$, $d = 0$, $m_x = 0.08196$, $s_x = 0.16217$, and $r = -0.2$. Current short rate is 6% and current market index is 370.

[5]. The rate here represents the fixed rate the AAA firm must receive to compensate its floating payments to an A rated firm. Other swap rates are interpreted similarly.

Line	Counterparty	Swap Rate	Spread	
(3)	AAA Firm	6.9718%	0.08 bp	(3) - (4)
(4)	Treasury	6.9710%		

Treasury Receives Fixed Rate				
Line	Counterparty	Swap Rate	Spread	
(5)	Treasury	7.0200%	0.01 bp	(5) - (2)
(6)			4.90 bp	(5) - (4)

The lower panel thus highlights the special feature of our swap pricing model. Our model takes the bilateral nature of default into account whereas the extant literature consider only one source of default at a time. As a result, the default risk that the A rated firm bore would be overpriced by 0.9 basis points for 2-year swap if the AAA rated firm did not consider the bilateral nature. When maturity extended to 10 years, this overpricing widened to 2.06 basis points.

V. VALUATION OF RISKY CALLABLE SWAPS AND SWAPTIONS

One of the main advantages of our swap pricing framework is the tractability of the model while we are tackling with three kinds of risks: interest rate risk and the default risks of two counterparties of the swap contract. The idea is simply to treat the two default risks from the viewpoint of the business cycle rather than treat them with two different firm value precesses, therefore reduce the dimension of the problem. Because of this, our model can be used to price those swap-related risky assets which involve the optimal exercise policy such as callable swaps and American swaptions. This is particularly important when one considers that the optimal exercise problem is exactly the limitation of the simulation approach.

Valuation of Risky Callable Swaps

A company sometimes not only wants longer-term protection against rising rates but also wants the flexibility to lock in a lower rate should rates decline. Similar to purchasing a callable bond, this company can purchase a swap

with an option to terminate that swap after a specific call date, for a fee paid upfront or on a spread basis.

In addition, a company that is uncertain of the term for which it will require fixed-rate funding can utilize callable swaps. Or, a bank that wants to fix the rate of funding a portfolio of mortgage-backed securities can use callable swaps to hedge against prepayment risk.

To obtain the value of a callable swap given fixed rate payment c, the same valuation equation (7) will be used, i.e.

$$\tfrac{1}{2}\sigma_r^2 r S_{rr} + \rho\sigma_r\sigma_x r^{\frac{1}{2}} x^{\gamma/2} S_{rx} + \tfrac{1}{2}\sigma_x^2 x^{\gamma} S_{xx} + [\kappa(\mu_r-r)-\lambda r] S_r + rx S_x + S_t - rS = 0,$$

The conditions $S(r,x,t;c)$ is subject to are equations (5), (6) and

$$S(r,x,t;c) \leq CP(t)$$

where $CP(t)$ is the price at which the swap is callable at time t. The equilibrium swap rate is again a c^* such that

$$S(r,x,0;c^*) = 0$$

Valuation of Swaptions

A swaption contract gives the investor a right to convert floating rate payments, $r(t)$, to fixed rate payments, c, on or before a predetermined date T. Let $S(r,x,t;c)$ be the swap value at time t, then the value of swaption, W, satisfies the following partial differential equation:

$$\tfrac{1}{2}\sigma_r^2 r W_{rr} + \rho\sigma_r\sigma_x r^{\frac{1}{2}} x^{\gamma/2} W_{rx} + \tfrac{1}{2}\sigma_x^2 x^{\gamma} W_{xx} + [\kappa(\mu_r-r)-\lambda r] W_r + rx W_x + W_t - rW = 0. \quad (8)$$

In addition, the option on swaps must also satisfy the following maturity condition,

$$W(r,x,T;c) = max(0, S(r,x,T;c)) \qquad (9)$$

and free boundary condition,

$$W(r,x,t;c) = S(r^*, x^*,t;c), \qquad (10)$$

where r^*, x^* are the endogenously solved critical interest rate and economic proxy at which the optimal early exercise takes place.

VI. CONCLUSION

This paper has introduced a new model for valuing corporate bonds. The model is then extended to price the default-risky interest swaps. Our model determines the default premiums for interest rate swaps relative to the current risk structure. The model's corporate bond prices of the swap counterparties are guaranteed to be those actually observed. The crux of the argument rests on modeling default behavior of a firm through the state of the economy and an implied default probability. The relative pricing for the default risk of derivatives appears to be an analog to the approach pioneered by Ho and Lee [13] who valued the default-free interest rate contingent claims relative to the initial term structure, however, the spirit of our model is consistent to the economic content of an equilibrium (partial) model.

Our model also can handle multiple sources of default risk with relative ease. Unlike the traditional arbitrage approach which typically deals with an extra source of default risk by expanding the dimension of state variables, the current model incorporates the extra source of default risk into boundary conditions of the valuation equation as we have demonstrated in the case of pricing default-risky swaps. Since we do not increase the dimension of the valuation equation, many other assets contingent on risky swap values and an optimal exercise policy such as callable swaps and swaptions can be priced under current framework.This paper may have significant contribution to financial research because it provides a procedure to price a broad range of default-risky contingent claims. Although in the paper we have treated the interest rate swaps and its related assets in particular, such assets as yield options, yield futures, caps, floors, and captions, etc., can all be valued with credit risk under the current framework. This paper also has important implications for the credit analysis practice in the financial industry and regulatory authority. Rather than striving for the value process of a firm, credit analysts and bank regulators typically look into the firm's quality rating, maturity, payment structure, and notional amount of the contract, etc. and then determine the default premiums according to a rule of thumb or simple simulations. This paper provides a theoretical foundation for such practice and can be used to evaluate the effectiveness of credit risk management, to assess the appropriateness of the capital adequacy criteria for commercial banks.

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APPENDIX

To begin with, we need to derive the dynamics of the value of a default-free discount bond. The instantaneous interest rate, r, follows a mean-reverting process

$$dr = \kappa(\mu_r - r)dt + \sigma_r r^{1/2} dz_r$$

and the value of a default-free discount bond, B, with maturity s is a function only of the interest rate and time. That is, $B = B(t,r;s)$. Apply Ito's Lemma, the instantaneous change in the value of such a bond is given by

$$dB = B_t dt + B_r dr + \tfrac{1}{2} B_{rr}(dr)^2$$
$$= [B_t + B_r \kappa(\mu_r - r) + \tfrac{1}{2} B_{rr} r \sigma_r^2] dt + B_r \sigma_r r^{1/2} dz_r$$

which can be rewritten as

$$dB \equiv B\mu_b(t,r;s)dt + B\sigma_b(t,r;s)dz_r, \tag{11}$$

where $\mu_b(t,r;s) = \{B_t + B_r \kappa(\mu_r - r) + \tfrac{1}{2} B_{rr} r \sigma_r^2 / B\}$ *and* $\sigma_b(t,r;s) = \{B_r \sigma_r r^{1/2} / B\}$.

Consider a zero investment portfolio formed by investing an amount W_i, $i=1,2$ in bond i with maturity s_i and borrowing an amount $W_1 + W_2$ at the instantaneous rate r. The instantaneous return of this portfolio is

$$W_1\{dB(t,r;s_1)/B(t,r;s_1) + W_2\{dB(t,r;s_2)/B(t,r;s_2) + (W_1+W_2)rdt$$

or

$$\{W_1[\mu_b(t,r;s_1)-r]+W_2[\mu_b(t,r;s_2)-r]\}dt + [W_1\sigma_b(t,r;s_1)+W_2\sigma_b(t,r;s_2)]dz_r$$

If W_1 and W_2 are chosen so that the return on the portfolio is non-stochastic, then to avoid riskless arbitrage the return must be zero. This implies that W_1 and W_2 should satisfy the following linear system

$$\begin{cases} W_1\sigma_b(t,r;s_1) + W_2\sigma_b(t,r;s_2) = 0 \\ W_1[\mu_b(t,r;s_1)-r] + W_2[\mu_b(t,r;s_2)-r] = 0 \end{cases}$$

If there is nontrivial solution of W_1 and W_2 for the above system, then it must hold:

$$[\mu_b(t,r;s_1)\text{-}r]/\sigma_b(t,r;s_1) = [\mu_b(t,r;s_2)\text{-}r]/\sigma_b(t,r;s_2)= \psi(t,r), \quad (12)$$

where $\psi(t,r)$ is the same for all bonds. That is, the rewards to variability for all default-free bonds are the same.

The interest rate model developed by Cox, Ingersoll, and Ross [8] begins with a detailed description of the underlying economy and derives the interest rate dynamics through a general equilibrium setting. The model is also able to give an exact form of the factor risk premium as below:

$$\psi = \lambda r^{1/2}/\sigma_r$$

Thus, Equation (12) can then be used to derive an equation for the price of a default-free discount bond. Writing the equation as

$$\mu_b(t,r;s)\text{-}r = (\lambda r^{1/2}/\sigma_r)\sigma_b(t,r;s)$$

and substituting for μ_b, σ_b yields, after rearrangement,

$$\tfrac{1}{2}\sigma_r^2 r B_{rr} + [\kappa(\mu_r\text{-}r)\text{-}\lambda r]B_r + B_t - rB = 0$$

With this result we are now prepared to derive the partial differential equation for the value of the risky debt by a similar argument. Dropping the maturity component s from equation (11) the instantaneous rate of return on any default-free discount bond is given by

$$dB = B\mu_b dt + B\sigma_b dz_r$$

Another source of uncertainty is the proxy of the economic condition which follows the CEV diffusion

$$dx = \mu_x x dt + \sigma_x x^{\gamma/2} dz_x. \quad (13)$$

Given these two sources of uncertainty, the value of a default-risky bond may be written as a function of these variables and time: $P(r,x,t)$. Then using Ito's Lemma, the instantaneous
rate of capital gain on the swap is given by

$$dP/P = \mu_p dt + (\sigma_r r^{\frac{1}{2}} P_r/P)dz_r + (\sigma_x x^{\gamma/2} P_x/P)dz_x \qquad (14)$$

where

$$\mu_p P = P_t + \kappa(\mu_r - r)P_r + \mu_x x P_x + \tfrac{1}{2}\sigma_r^2 r P_{rr} + \rho r^{\frac{1}{2}} x^{\gamma/2} \sigma_r \sigma_x P_{rx} + \tfrac{1}{2}\sigma_x^2 x^{\gamma} P_{xx}$$

and ρ is the instantaneous correlation between dz_r and dz_x.

Consider forming a zero net investment portfolio by investing amounts W_p, W_b, W_x in the corporate bond, the default-free bond and the market portfolio respectively, and borrowing $W_p + W_b + W_x$ at the instantaneous rate r. The instantaneous return on this portfolio is then, using (11), (13), and (14),

$$W_p \cdot dP/P + W_b \cdot dB/B + W_x \cdot dx/x - (W_p + W_b + W_x)rdt$$
$$= [W_p(\mu_p - r) + W_b(\mu_b - r) + W_x(\mu_x - r)]dt + [(W_p \sigma_r r^{\frac{1}{2}} P_r/P) + W_b \sigma_b]dz_r$$
$$+ [(W_p \sigma_x x^{\gamma/2} P_x/p) + W_x \sigma_x x^{\gamma/2 - 1}]dz_x$$

Let

$$W_p(\sigma_r r^{\frac{1}{2}} P_r/P) + W_b \sigma_b = 0 \qquad (15)$$

$$W_p(\sigma_x x^{\gamma/2} P_x/P) + W_x \sigma_x x^{\gamma/2 - 1} = 0 \qquad (16)$$

then the instantaneous rate of return on the portfolio is certain, and to exclude riskless arbitrage it must be equal to zero so that

$$W_p(\mu_p - r) + W_b(\mu_b - r) + W_x(\mu_x - r) = 0 \qquad (17)$$

Solve W_b, W_x from (15), (16), and substitute them into equation (17). Eliminating W_p and using the definitions of μ_p and λ, we obtain

$$\tfrac{1}{2}\sigma_r^2 r P_{rr} + \rho \sigma_r \sigma_x r^{\frac{1}{2}} x^{\gamma/2} P_{rx} + \tfrac{1}{2}\sigma_x^2 x^{\gamma} P_{xx} + [\kappa(\mu_r - r) - \lambda r]P_r + rx P_x + P_t - rP = 0$$

which is equation (3) of the text.

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