INTERNATIONAL JOURNAL OF BUSINESS, 8(4), 2003 ISSN:1083-4346

Portfolio Insurance Strategies: A Comparison of Standard Methods When the Volatility of the Stock Is Stochastic

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ABSTRACT

We compare the performances of the two standard portfolio insurance methods: the Option Based Portfolio Insurance (OBPI) and the Constant Proportion Portfolio Insurance (CPPI), when the volatility of the stock index is stochastic. In this framework, we provide a quite general formula for the CPPI portfolio value. We use criteria such as comparison of payoffs functions at maturity and various quantiles. We emphasize in particular the role of the insured percentage of the initial investment.

JEL: *G0, G15*

Keywords: *Portfolio insurance; OBPI; CPPI; Stochastic volatility*

I. INTRODUCTION

One of the more popular strategies of portfolio insurance is the Option Based Portfolio Insurance (OBPI), introduced in Leland and Rubinstein (1976). It consists basically in buying simultaneously the stock (generally a financial index) and a put written on it. The value of this portfolio at maturity is always greater than the strike of the put, whatever the market fluctuations. Thus, this strike is the insured amount, which is often equal to a given percentage of the initial investment.

The CPPI method has been analysed in Black and Rouhani (1989) and Black and Perold (1992). This method is based on a particular strategy to allocate assets dynamically over time. The investor starts by choosing *a floor* equal to the lowest acceptable value of the portfolio. Then, he computes *the cushion* that is equal to the excess of the portfolio value over the floor. Finally, the amount allocated to the risky asset (usually called *the exposure*) is determined by multiplying the cushion by a predetermined multiple. The remaining funds are invested in the reserve asset (for example, Treasury bills or other liquid money market instruments.) Initial cushion, multiple, floor and tolerance can be chosen according to the investor's own objective.

The higher the multiple, the more the portfolio value increases in a bullish market. Nevertheless, the higher the multiple, the nearest the portfolio will be to the floor in a bearish market. As the cushion approaches zero, the amount invested on the risky asset approaches zero too. This feature implies that the portfolio value is below the floor only when there is a very sharp drop in the market before the investor can modify his investment weights. Therefore, the multiple must be bounded as shown in Bertrand and Prigent (2002a). Thus, this strategy is rather simple with respect to other approaches.

The purpose of this paper is to compare these two strategies of portfolio insurance, when the volatility of the risky asset is stochastic. In section 2, we recall the basic properties of these two strategies. In section 3, we examine the impact of stochastic volatility for each portfolio insurance method. Finally in section 4, we analyse their properties and compare them. For this purpose, first we examine their payoffs and compute their expectations, variances, skewness and kurtosis of their returns. Second, we evaluate some of the quantiles of their returns.

II. BASIC PROPERTIES OF THE OBPI AND THE CPPI

We consider the following financial market: the period of time considered is $[0,T]$. Denote B_t the riskless asset which has the following dynamics: $dB_t=B_t\, r\, dt$. Assume that the risky asset S_t is a diffusion process: $dS_t=S_t\, [a\, dt + \sigma_t\, dW^1_t]$, where $(W^1_t)_t$ is a standard Brownian motion.

The volatility σ_t is assumed to be stochastic and is defined as solution of the following stochastic differential equation:

$$d\sigma_t = \alpha\,(t, \sigma_t)dt + \beta\,(t, \sigma_t)\, dW^2_t,$$

where $(W^2_t)_t$ is another standard Brownian motion independent from $(W^1_t)_t$.

We particularize the case of a stochastic volatility that evolves according to an Ornstein-Uhlenbeck process, as introduced in Scott (1987) and Stein and Stein (1991):

$$d\sigma_t = k(\sigma^* - \sigma_t)dt + \beta\, dW^2_t,$$

where σ^* is the long-run average level of σ, the non-negative parameter k determines the speed of convergence of the volatility σ_t to σ^*. The parameter β is the volatility of the volatility σ_t. Recall the explicit formula of the volatility:

$$\sigma_t = \sigma_0.e^{-k.t} + \sigma^*.\left(1 - e^{-k.t}\right) + \beta\int_0^t e^{-k.(t-u)} dW_u^2$$

We now define the two insurance portfolio strategies.

For the OBPI method, introduce the portfolio value V^{OBPI} which is defined at the terminal date T by:

$$V^{OBPI}(T) = S(T) + [K - S(T)]^+$$

By the Put/Call parity, we obtain also:

$$V^{OBPI}(T) = S(T) + [S(T) - K]^+$$

The value of the portfolio is not only insured at the final date T but also at any time of the portfolio management period: by no-arbitrage argument, there exists a deterministic level of insurance equal to $Ke^{[-r(T-t)]}$.

Examine now the CPPI method. We assume that the floor F_t is not stochastic. Thus F_t follows the dynamic: $dF_t = F_t\, r\, dt$. The value of the initial floor F_0 must be less than the initial portfolio value $V^{CPPI}(0)$. Let $V^{CPPI}(t)$ and C(t) be the values of the portfolio and of the cushion respectively. By definition, the cushion is equal to the difference between the portfolio's and the floor's values: $C_t = V_t - F_t$.

Denote by e_t the exposure. It is the total amount invested in the risky asset. The standard CPPI method consists in letting $e_t = mC_t$ where m is a constant called the multiple. This latter parameter is always assumed to be greater than 1, in order to get a convex payoff function. This last condition allows to make profit of stock price increments in bullish markets.

Thus, the OBPI has just one parameter, the strike K of the put. The CPPI method is based on the choice of two parameters: the initial floor F_0 and the multiple m. Therefore, the strike K plays the same role as $F_0\, e^{[rT]}$ in the CPPI model.

Assume that the decision criteria of the investor is the amount insured, K, at maturity T. In order to compare the two methods, we first assume that the initial amounts V_0^{OBPI} and V_0^{CPPI} are equal. Secondly, we assume that they provide the same guarantee K at the final date. Consequently, we have to choose $F_0 = Ke^{(-rT)}$, in order to obtain $F_T = K$.

III. EFFECT OF STOCHASTIC VOLATILITY ON BOTH PORTFOLIO INSURANCE METHODS

We illustrate the impact of stochastic volatility on both portfolio insurance methods. We set the constant volatility in the Black-Scholes model equal to the long-term volatility of the stochastic volatility model.

A. The OBPI Case

To conduct the comparison, consider three strikes *K*: in, at and out the money. For each strike *K*, the insured amount in the stochastic volatility case is equal to $q(K).V_0^{OBPI}$.

Black-Scholes Call prices, $C^{BS}(K)$, are obviously different from stochastic volatility Call prices, $C^{SV}(K)$. Therefore, in order to keep the same initial portfolio value, V_0^{OBPI}, and the same insured amount, we have to adjust the quantity invested on the riskless asset and on the call. For the stochastic volatility case, we choose to normalize it to 1. Denote by n the quantity to invest on the call in the Black-Scholes model. The parameter n is determined from the following relation:

$$V_0^{OBPI} = q(K).V_0^{OBPI}.e^{-r.T} + C_0^{BS}.(K')$$

where $K' = \frac{q(K).V_0^{OBPI}}{n}$ is the new strike in the Black-Scholes model. Since, as usual, the call price $C^{SV}(K)$ for the stochastic volatility is higher than the Black-Scholes price, $C^{BS}(K)$, we deduce that the strike K' is smaller than the strike K.

We illustrate the comparison for the following numerical base case: a= Ln(1.1)=9,53 %, r=3%, T=1, k=0.7, $\sigma^* = 20.19\%$, $\beta = 0.5$.

Table1

Correspondence between BS and SV models

K	V_0^{OBPI}	q(K)	n	K'
110	112,75	97,56%	0,9823	108,05
100	107,30	93,2%	0,9850	98,51
90	103,74	86,75%	0,9856	88,70

Using the preceding values, we can compare the first four moments of the OBPI return at maturity for BS and SV models. In the following table, we report results for the at-the-money case (K=100).

Table 2
Comparison of the first four moments of the rate of return

	OBPI BS	OBPI SV
Expectation	7,26%	7,79%
Standard Deviation	16,66%	21,43%
Relative skewness	1,45	4,29
Relative kurtosis	5,43	75,27

The OBPI SV exhibits higher expected return and skewness but also higher standard deviation and kurtosis. The same qualitative results hold for the in and out-the-money cases.

Changing constant volatility for stochastic volatility induces two opposite effects:

- The reduction of the call payoff since the strike increases ($K>K'$).
- The rise of the probability to exercise the call since the relative skewness of the risky asset increases.

The second effect has a bigger impact on the return of the OBPI than the first one. This explains why we recover the common features of stochastic volatility models despite the variation of the strike.

B. The CPPI Case

We conduct comparisons of the CPPI BS and SV returns for the three previous initial portfolio values (see Table 3).

Table 3
Comparison of the first four moments of the rate of return

	m=4		m=6		m=8	
	CPPI BS	CPPI SV	CPPI BS	CPPI SV	CPPI BS	CPPI SV
Expectation	5,98%	5,96%	7,69%	7,64%	9,81%	9,78%
Standard Deviation	12,25%	14,44%	25,39%	34,52%	55,49%	86,14%
Relative skewness	3,58	7,27	8,22	30,59	22,13	68,19
Relative kurtosis	29,7	148,6	154,7	2461	1039	9222

For the CPPI, the expectation of the BS model is slightly higher than that of the SV model, for the three values of the multiple m. This property can be related to the negativity of the vega of the CPPI (derivative of the CPPI value with respect to the constant volatility) as shown for example in Bertrand and Prigent (2002b). As expected, the stochastic volatility increases the three other moments.

IV. COMPARISON OF THE PORTFOLIO VALUES AT MATURITY

A. Comparison of Payoffs

For the standard case (i.e. when the stock price follows a geometric Brownian motion), we can compare the terminal payoffs of the two strategies as functions of *S(T)*, for all values of the multiple *m* greater than one and for strikes *K* at, in and out the money. For both the CPPI and the OBPI, we set the same initial value of the portfolio. We deduce the two payoffs at maturity *T*. For the OBPI, the time *T* value of the strategy is defined by:

$$V^{OBPI}(T)=K+(S(T)-K)^+$$

For the CPPI, the time *T* value of the strategy is defined by:[1]

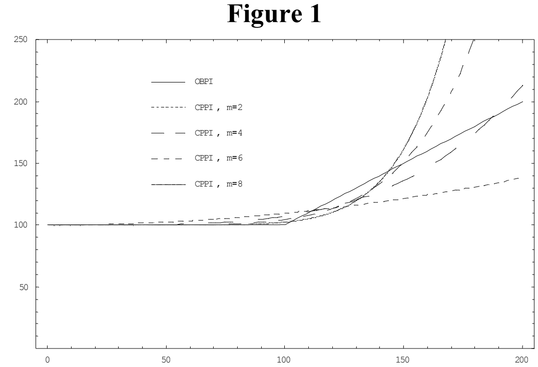
$$V^{CPPI}(T)=K+\alpha_T S(T)^m$$

The parameter α_T is a constant with respect to *S(T)*. Thus, the value of the CPPI portfolio at maturity is a convex function in *S(T)* as soon as the multiple *m* is greater than *1*, which is the usual assumption.

Due to the absence of arbitrage, none of the two payoffs is greater than the other, for all terminal values of the risky asset. The two payoff functions intersect one another.

To illustrate what happens, consider the following figure.

Figure 1



We can check on this example that the two curves intersect one another for the different values of m considered (*m=2*, *m=4*, *m=6* and *m=10*). For moderate values of the risky asset above the strike, the OBPI payoff is above the CPPI payoff. The converse is true for high values of the risky asset or values below the strike.

When the volatility of the stock price is stochastic, the financial market is incomplete and we have to select a particular risk-neutral probability. In what follows, we choose the minimal one, previously introduced in Föllmer and Schweizer (1991). It corresponds to setting the risk premium on the volatility to zero, as assumed for example in Hull and White (1987). The payoff at maturity associated to the OBPI method is still a function of *S(T)*, defined by:

$$V^{OBPI}(T)=K+(S(T)-K)^{+}$$

For the CPPI, the time T value of the strategy is now defined by:[2]

$$V^{CPPI}(T)=K+\alpha_T S(T)^m,$$

where the parameter α_T is no longer a constant but a random variable depending on the average of the cumulative squares of the volatility, denoted by ν_T with:

$$\nu_T = \frac{1}{T}\int_0^T \sigma_s^2 ds$$

Again, due to the absence of arbitrage, none of the two payoffs is greater than the other, for all terminal values of the risky asset. As shown in Appendix 3, the expectation of ν_T is higher than in the Black and Scholes' model. It explains in particular why for the CPPI, the expectation of the BS model is higher than that of the SV model (see Table1).

B. Comparison of the Expectation, Variance, Skewness and Kurtosis.

When dealing with options, the mean-variance approach is not always justified since payoffs are not linear. So we examine simultaneously the first four moments. If we compare the first two moments (mean-variance analysis), note that for *m* high, the expectation and variance of the CPPI portfolio are greater than those of the OBPI one and so there is no-dominance with respect to the mean-variance criterion. For any parametrization of the financial markets, there exists at least one value for *m* such that the OBPI strategy dominates, in a mean-variance sense, the CPPI one.

The following example gives an illustration with the same values for the parameters as previously. We choose *m* such that

$$E[R^{OBPI}(T)]=E[R^{CPPI}(T)],$$

we find *m*=5.5137.

For that value of *m*, CPPI strategy is dominated, in a mean-variance sense, by OBPI strategy. Nevertheless, if we take into account the relative skewness, we find that CPPI strategy is much skew to the right that OBPI strategy. But the probability of extreme events, as shown by relative kurtosis, is much bigger in the case of the CPPI strategy.

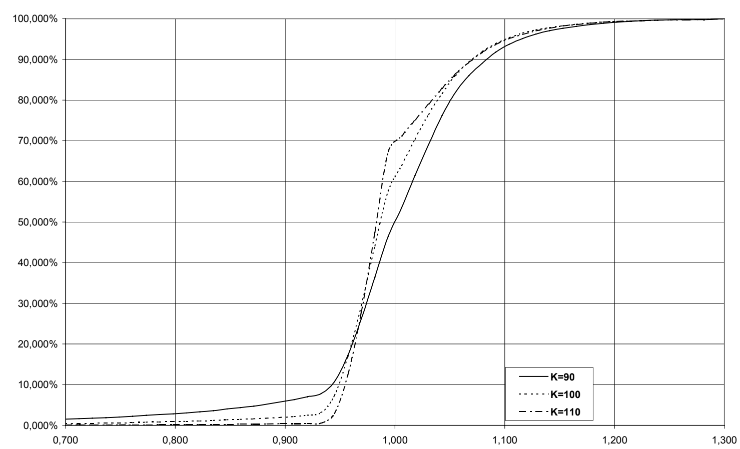
Table 4
Comparison of the first four moments of the rate of return

	OBPI	CPPI
Expectation	7,79%	7,79%
Volatility	21,43%	27,73%
Relative skewness	4,29	11,558
Relative kurtosis	75,27	250,529

C. Comparison of Quantiles

We evaluate the probability that the CPPI portfolio value is greater than that of the OBPI. We need to compute the cumulative distribution function of the ratio: $\frac{V_T^{OBPI}}{V_T^{CPPI}}$

Figure 2



This figure shows that:

- For *K=90* (in the money call), the probability that the CPPI portfolio value is higher than the OBPI one is approximatively 0.5. Moreover, the CDF is roughly symetric around 1.

- As soon as K rises (as the percentage of the initial insured investment rises), this is no longer true. The probability that the CPPI portfolio value is higher than the OBPI one is approximatively 0.6 for $K=100$ and 0.7 for $K=110$. This arises because the probability of exercising the call decreases with the strike.
  This finding has important practical implications as it means that the CPPI seems to be more desirable in probability as the insured percentage of the initial investment increases.

V. CONCLUSION

We have examined the two main portfolio insurance methods: the Option Based Portfolio Insurance and the Constant Proportion Portfolio Insurance. In particular we have focused on the introduction of stochastic volatility in the dynamics of the risky asset. We have first analyzed each method in a Black-Scholes world and in a stochastic volatility world in which the volatility is modelled by an Ornstein-Uhlenbeck process. We have shown that on one hand stochastic volatility increases the return of OBPI while reducing slightly the return of the CPPI. On the other hand, stochastic volatility increases the standard deviation, the skewness and the kurtosis of the portfolio returns for both methods. Therefore, CPPI performances are more affected by stochastic volatility than OBPI ones. We then consider the comparison of the two methods in the stochastic volatility framework. We show in particular that as soon as the percentage of the initial insured investment rises, the probability that the CPPI portfolio value is higher than the OBPI increases. This property has to be noted for guaranteed fund management.

NOTES

1. For this result, we refer to Prigent (2001) for example. A generalization is given in Appendix 1 and 2.
2. For details about the determination of this formula, see Appendix 1 and 2.

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APPENDIX 1
Determination of the cushion

We provide the determination of the CPPI portfolio value. It is solution of the following equation:

$$dV_t = (V_t - e_t)\frac{dB_t}{B_t} + e_t \frac{dS_t}{S_t}$$

Recall that: $V_t = C_t + F_t$ and $e_t = mC_t$

Therefore, the cushion is determined from the relation:

$$dC_t = (F_t + (1-m)C_t)\frac{dB_t}{B_t} + mC_t \frac{dS_t}{S_t}$$

Since the floor satisfies: $dF_t = rF_t dt$, we deduce that:

$$dC_t = C_t\left[(1-m)\frac{dB_t}{B_t} + m\frac{dS_t}{S_t}\right]$$

Now, using the dynamics of the risk free asset and of the risky asset, we obtain:

$$dC_t = C_t\left[(r + m(a-r))dt + m\sigma_t dW_t^1\right]$$

Consequently, using the stochastic exponential formula, we get the cushion value at any time t:

$$C_t = C_0 \exp\left[(r + m(a-r))t - \frac{1}{2}m^2\int_0^t \sigma_s^2 ds + m\int_0^t \sigma_s dW_s^1\right] \tag{A1}$$

Note that this formula is quite general since it is true for all standard stochastic volatility models.

APPENDIX 2
The cushion as a function of the risky asset price

The risky asset price with stochastic volatility is given by:

$$S_t = S_0 \exp\left[at - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s^1\right]$$

Therefore:

$$S_t^m = S_0^m \exp\left[m\left(at - \frac{1}{2}\int_0^t \sigma_s^2 ds \right) + m\int_0^t \sigma_s \, dW_s^1 \right] \tag{A2}$$

Using relations (A1) and (A2), we obtain: $C_t = S_t^m . \alpha_t$ with

$$\alpha_t = \left(\frac{C_0}{S_0^m} \right) \exp[\beta_t . t], \quad \text{where } \beta_t = -(m-1).r - \frac{1}{2}\left(m^2 - m\right)\frac{1}{t}\int_0^t \sigma_s^2 ds \tag{A3}$$

This relation is also quite general since it does not depend on particular assumptions on the volatility dynamics, except that we suppose that this volatility has no jump (as for all standard stochastic volatility models).

APPENDIX 3

Expectation of ν_t

This relation is important to measure the impact of the stochastic volatility. We obtain:

$$E(\nu_t) = \frac{1}{T}\int_0^T E\left(\sigma_t^2\right)dt = (\sigma *)^2 + (\sigma_0 - \sigma *)^2 \left[\frac{1 - e^{-2kt}}{2kt} \right] + 2\sigma * (\sigma_0 - \sigma *) \left[\frac{1 - e^{-kt}}{kt} \right]$$
$$+ \frac{\beta^2}{2k}\left[1 - \frac{1 - e^{-2kt}}{2kt} \right]$$

Therefore, the expectation of the cumulative volatility squares is higher than in Black and Scholes' model.