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Flexible-Rate Mortgages

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ABSTRACT

We propose a model for the endogenous determination of an optimal refinancing policy for mortgage loans under limited refinancing opportunities. Transaction costs are also included in the analysis. A detailed examination of the optimal exercise distributions sheds light on the impact of contract features on the average prepayment behavior in mortgage-backed securities.

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[*] The present article extends and superseded "Roncoroni, A., 2000, 'The S Option.' Working Paper, University Paris Dauphine."

I. INTRODUCTION

A mortgage is a loan secured on real estate. The borrowed amount, named principal, is repaid along a time horizon through installments. These payments are usually computed pro-rata temporis, meaning that each cash flow splits in two components: a first component represents a portion of the outstanding balance; a second component is the interest accrued since the last payment. This scheme guarantees that the loan is totally repaid by the end of the contract. Most of the residential mortgage loans offered to retail investors are expected to amortize through a French amortization scheme consisting of constant installments, usually paid off on either a monthly or quarterly basis.

In the U.S. market, the vast majority of mortgages are collected into pools managed by government-sponsored agencies. These institutions issue securitized notes backed by the cash flows generated by a specific pool of mortgages and sell these assets, known as mortgage-backed securities (MBS), to private investors, generally large investment funds, either directly of through dealers.

In principle, the value of an MBS is the value of a long-term annuity with fixed maturity. However, the borrower is allowed to pre-pay his debt back any time before the legal maturity of the loan. Prepayment may occur due to either exogenous reasons such as moving or any other personal issue, or endogenous reasons typically linked to the mortgagor's ability to enter a new mortgage at more favorable rate conditions. This usually occurs as market lending rates decrease and the borrower has the opportunity to get profit from such circumstances. In practice, market rates may fall and the standing mortgage is still continued. This is the case of a mortgagor whose credit status has deteriorated and who is thus required to have a greater spread over the market lending rate. Transaction fees applying to a mortgage prepayment constitute a further incentive to continuing the standing contract while market rates decrease. In actuality, an MBS can be seen as a long position in a fixed-maturity annuity and a short position in a compound American-style call option.

The literature on the valuation of MBS's splits into two distinct frameworks. Early works by Dunn and McConnell (1981a, b) and Brennan and Schwartz (1985) propose a rational model explaining how a mortgage borrower chooses to refinance his loan. They determine the fair value of an MBS by applying contingent claim valuation to the portfolio consisting of a long annuity and a short American-style option under the hypothesis that the option value is maximized. Notice that this is equivalent to assuming that the mortgagor minimizes the value of his standing mortgage position.

A second framework for valuing MBS's builds on the econometric identification of the prepayment behavior from historical data. Models within this setting have been proposed by Schwartz and Torous (1989, 1992) and Boudoukh et al. (1997) and now constitute the standard market practice for valuing MBS's. This setting suffers from a major drawback: they perform quite badly in out-of-sample predictions. This is mainly due to a strong dependence on market conditions producing the historical data on which econometric analysis is conducted. Stanton (1995) develops an alternative model combining elements from the econometric and the rational prepayment approaches.

The present work considers the mortgage refinancing problem as seen from the mortgagors' viewpoint, much in the spirit of the first line of research mentioned above. We determine the optimal prepayment rule for a borrower minimizing the value of his mortgage position by choosing when the mortgage is to be refinanced. Following the theoretical framework introduced in Roncoroni (2000), we add a constraint on the number of refinancing options. This feature accommodates the possibility that the lender gives the borrower the option to refinance his mortgage internally at smaller additional costs than the ones presented upon repaying the principal and entering into a new mortgage with another lender (e.g., transaction costs and credit spread variations). This option represents a way to attract the customer's fidelity to the lending institution and let it save all costs required to search for alternative investment opportunities. The optimal recursive determination of prepayment policies has recently been investigated by Longstaff (2002), Gocharov and Pliska (2003) and Kalotay, Yang, and Fabozzi (2006) under unconstrained refinancing opportunities.

We suppose that refinancing is subject to small fees or transaction costs. For simplicity we focus on refinancing decisions exclusively steered by better market conditions for the borrower. The inclusion of exogenous elements driving the refinancing policy does not pose any particular problem to the proposed setting. It suffices to allow the time horizon to be dependent on the occurrence of the random events triggering the contract expiration. The number N of refinancing opportunities over the time horizon is fixed at the outset. At each point in time the mortgagor sets the debt rate to either the current rate or to newly available floating lending rate. Over the contract lifetime, this option can be exercised N times at most.

From a financial viewpoint, the mortgagor's decision problem takes the form of a multiple compound American-style option. This allows us to use stochastic dynamic programming methods to tackle the determination of the optimal prepayment policy defined as the one minimizing the value of the contract.

The paper is organized as follows. Section 2 describes the contract feature of fixed-rate and floating rate mortgages under limited refinancing options. Section 3 details the solution methodology and extends the analysis to the case including transaction costs. Section 4 illustrates empirical results obtained by performing experiments under alternative scenarios. A final section presents a few concluding remarks.

II. THE MODEL

A. Fixed-Rate Mortgage

We consider a finite time horizon $T_{0,T}$:=(0,1,...,T). At time 0, an individual borrows 1 Euro. This generates a balance due B(0)=1. The borrower is required to pay back this amount together with interest over the horizon according to a constant installment scheme. For each period [t–1,t] an interest I(t–1) is calculated on the outstanding balance B(t–1) at a rate equal to r(t–1). The outstanding exposure B′(t–1) of the debtor over the period [t–1,t] is defined by the due interest I(t–1) plus the standing balance B(t–1). At the end of the period, i.e., at time t, the borrower pays a fraction f(t) which is written down in relation to the standing exposure B′(t–1). This number is computed as a proportion of the residual contract lifetime, namely f(t) = 1/(T–(t–1)). Let us denote this payment by P(t–1). The explicit dependence on the starting day t–1 of the period underlines that this amount is set at this time, though paid off at the end of the period. After performing this payment, the new outstanding balance for the debtor becomes B(t) = B′(t–1) – P(t–1). Table 1 summarizes the steps involved in this payment scheme. It is clear that the new balance due depends on the initial balance and on the debt rate process r = (r(t), t=0, ..., T–1). Since by hypothesis the initial debt position is B(0)=1, the only exogenous ingredient is the debt rate process r.

Table 1
The amortization scheme

Quantity	Symbol	Formula
Standing debt balance at t – 1	B (t – 1)	given by induction
Debt rate for period [t – 1, t]	r (t – 1)	Random
Interest accrued on [t – 1, t]	I (t – 1)	r (t – 1) × B (t – 1)
Standing exposure on [t – 1, t]	B' (t – 1)	I (t – 1) + B (t – 1)
Constant installment coefficient	f (t)	1 / (T – (t – 1))
Payment to lender at time t	P (t – 1)	f (t) × B' (t – 1)
Standing debt balance at time t	B (t)	B' (t – 1) – P (t – 1)

Notice that the borrowed capital is totally repaid by the end of the time horizon, i.e., B(T) = 0. This can be proved by showing that the last payment P(T–1) matches the standing balance at time T–1:

$$P(T-1) = B'(T-1)/(T-(T-1) = B'(T-1).$$

The *cost to go* associated to a given debt rate process r is defined as the sum of all cash flows stemming from the payment scheme just described, i.e., $\sum_{t=0}^{T-1} P(t)$. We now move to the description of the debt rate process r.

B. Flexible-Rate Mortgage

Let us assume that the market quotes an interest rate R and suppose this number follows a time-homogeneous Markov chain R=(R(t),t=0,...,T) with finite state space $S = \{s_{min} + k\Delta s, k = 0,1,...,K\}$. Here the minimum rate s_{min}, the interest lag Δs and the cardinality K+1 are all fixed. Let $S' = \{s_{min} + k\Delta s, k = 1,..., K-1\}$. The transition probabilities of the process R are assigned as follows:

$$p(x,y)=\begin{cases}\frac{1}{3} & \text{if } (x,y)\in S'\times\{x, x\pm\Delta s\},\\ \frac{1}{2} & \text{if } (x,y)\in\{S_{min}\}\times\{S_{min}, S_{min}+\Delta s\}\cup\{S_{max}\}\times\{S_{max}, S_{max}-\Delta s\},\\ 0 & \text{otherwise.}\end{cases}\quad (1)$$

This function, coupled with an initial market rate R(0), induces a probability measure $P_{R(0)}$ on the path space S^{T+1}.

Turning to the debt rate process r, we suppose its initial value r(0) matches the market rate R(0) at the same time. At any date t, the debtor is faced with the choice of continuing the mortgage at the standing debt rate r(t–1) or to repay the entire due capital by using the proceeds from entering a new mortgage at the currently available conditions expressed by the prevailing market rate R(t). In this case, we say that the mortgage has been "refinanced".

We study the case where this option can be exercised a maximum number N of times over the horizon (0,...,T–1). If N is equal to the number T–1 of setting times, the optimal strategy for the borrower is trivial: he exercises the option each time the market rate R goes down. Consequently, we suppose that the number N of refinancing opportunities is strictly smaller than the number of dates. Table 2 indicates the input parameters and dynamic processes defining the contract provisions.

Table 2
Input variables and parameters

Quantity	Symbol	Formula
Time horizon	$T_{0,T}$	{0,1,...,T}
Market rate range	S	$\{s_{min}+k\Delta s, k=0,1,...,K\}$
Initial market rate	R(0)	Given
Market rate dynamics	R	(R(t),t=0,...,T) ~ (1)
Number of options	n(0)	$N<T$
Initial debt rate	r(0)	R(0)
Balance due	B(0)	1

Turning to the mortgage refinancing decision, this can be described by a process α specifying at each time t=0,...,T–1 whether the mortgagor continues his position at the standing conditions (α = 0) or, if possible, refinances his debt (α = 1). The control process is described in Table 3.

Of course, the chosen control policy affects the state variables dynamics featuring the borrower's position over time. These variables are as follows: 1) the number $n^\alpha(t)$ of available refinancing opportunities left for future exercise; 2) the current interest rate $r^\alpha(t)$; 3) the installment cash flow $P^\alpha(t)$ for the current period; 4) the resulting outstanding balance $B^\alpha(t)$. Table 4 reports all processes subject to a control. It is clear that choosing to refinance or not at a given time t has an effect on $n^\alpha(t+1)$ and $r^\alpha(t+1)$.

Table 3
Control variables

Quantity	Symbol	Formula
Control policy process	α	$(\alpha(t), t=0, ..., T-1)$
Control policy domain	$D(\alpha(t))$	$\begin{cases} \{0,1\} \text{ if } \sum_{i=1}^{t} \alpha(i) < n(0) \\ \{0\} \text{ otherwise} \end{cases}$
Control policy at time t	$\alpha(t)$	$\begin{cases} 0 \text{ (cont.)} \\ 1 \text{ (refin.)} \end{cases}$

Table 4
Controlled system

Quantity	Symbol	Formula
Refinancing options process	n^α	$(n^\alpha(t), t=0,...,T-1)$
Refinancing opportunities at t	$n^\alpha(t)$	
Controlled debt rate process	r^α	$(r^\alpha, t=0,...,T-1)$
Debt rate at time t	$r^\alpha(t)$	$\begin{cases} R(t), \text{if } \alpha(t)=1 \\ r^{\alpha(t-1)}(t-1), \text{ if } \alpha(t)=0 \end{cases}$
Pro-rata payment process	P^α	$(P^\alpha(t), t=0,...,T-1)$
Standing debt balance process	B^α	$(B^\alpha(t), t=0,...,T-1)$

Table 5
Debt process

$B^\alpha(t)$	Outstanding debt balance at time t
$B'(t) = B^\alpha(t)\left(1+\frac{r^\alpha(t)}{52}\right)$	Capital plus interest on [t,t+1]
$P^\alpha(t) = \frac{B'(t)}{T-t}$	Pro-rata payment for [t,t+1]
$B^\alpha(t+1) = B'(t) - P^\alpha(t)$	Outstanding debt balance at time t+1

The link to the other two quantities is clarified in Table 5. The borrower wishes to select a control policy that minimizes the cost associated with the entire repayment stream ($P^{\alpha}(t)$,t=0,...,T–1). For the purpose of illustrating this issue, we consider the sum of all these payments as a measure of this cost. Notice that we do not consider discounting for evaluation purposes.

The problem is qualified as follows:

$$\min_{\alpha \in A} E\left(\sum_{j=0}^{T-1} P^{\alpha}(j) \right), \tag{2}$$

where A denotes the class of admissible control policies $\alpha=\{\alpha_0,...,\alpha_{T-1}\}$ determining the controlled payment process ($P^{\alpha}(t), 0 \le t < T$)[1].

We note that any control policy α defines a multivariate stopping rule ($\tau_1, ..., \tau_n$) ($n \le N$) which is recursively defined by

$$\tau_1 = \inf\{t \ge 0 : \alpha(t)=1\},$$
$$\tau_{k+1} = \inf\{t > \tau_k : \alpha(t)=1\}.$$

Our task is to build a model for describing the financial structure illustrated above, then provide an algorithm delivering the minimum value in (2), and finally determine the corresponding optimal control policy $\alpha^*=(\alpha^*(0), ..., \alpha^*(T-1))$. This latter is equivalent to the multivariate optimal stopping rule $\left(\tau_1^*, ..., \tau_N^*\right)$. We end this section by deriving an expression for the time t+1 outstanding balance in terms of the adopted control policy and the debt level recorded one time-step before:

$$B(t+1) = B'(t) - P(t) = B'(t) - \frac{B'(t)}{T-t} = c(r^{\alpha(t)}(t))B(t)T(t)$$

where $c(x)=1+x/52$ denotes the accrual factor corresponding to rate x and T (t) is the *pro-rata* coefficient defined by $1-1/(T-t)$.

III. Optimal Exercise Policy

A. Markov Control Policies

Recall that the state variable of a Markov control problem is defined as the information upon which the control policy is chosen at any time. More precisely, this is the set of observable variables upon which α(t) can be determined at a given point in time.

We observe that the time t standing balance B(t) and debt rate r(t) fully determine both the updated capital-plus-interest B′(t) and the constant payments P(t), which in turn goes into the objective functional (2). Since the target is affected by both B(t) and r(t), the time t control α(t) should depend on these variables. The dependence on B(t) is straightforward. If any refinancing opportunity is still available, i.e., $0<n(t) \le N$, then the

debt rate is set to the best performing one between its previous value r(t–1) and the current market rate R(t) observed in the market place at the same time. If instead all refinancing options have already been exercised, i.e., n(t)=0, then the new current debt rate r(t) must agree with its previous value r(t–1). In general, r(t), and thus a Markov control policy α(t), depends on the triplet (R(t), r(t–1),n(t)).

These considerations lead to consider control policies whose time t value depends on the outstanding balance B(t), the market rate R(t), the one-period-ahead debt rate r(t–1) and number n(t) of available refinancing options:

$$\alpha(t) = F(t, B(t), R(t), r(t-1), n(t))) \tag{3}$$

If B, R, r, n represent possible values taken by B(t), R(t), r(t), and n(t), respectively, the 4-uple (B,R,r,n) is a candidate state variable.

We adopt the dynamic programming principle for the purpose of computing the value function and the optimal exercise policy over the contract lifetime. Since the state variable B(t) is continuous, a direct application of this principle is prevented. However, the value function is homogeneous of degree one in this variable and the value function can be written as

$$V(t,(y,x,r,n)) = y \times V^*(t,(x,r,n)) \tag{4}$$

We refer to V* as the Unitary Value Function because it represents the value function per unit of outstanding balance. Since y is non-negative, the optimal policy for V coincides with the optimal policy for V*. As a consequence, admissible stopping rules are independent of y. We therefore consider control variables of the form

$$\alpha(t) = F(t,(R(t), r(t-1), n(t))) \tag{5}$$

The control variable so defined is F_t-measurable, implying that the control policy (5) is admissible. The state variable of our problem is the triple (R,r,n) defined on a subset of $X = S \times S \times \{0,...,N\}$. For instance, if t=1, $r \in \{r_0, r_0+\Delta s, r_0-\Delta s\} \subset S$. It turns out that the loss in terms of computational complexity for restricting numerical calculations to the exact domain of the state variable highly overcomes the gain resulting from reducing the number of computations. Consequently, we decide to skip considering the domain constraints while performing the optimization algorithm and compute the value function over the entire domain X.

B. Dynamic Programming Algorithm

Having identified the state variable, we now turn to the recursive computation of the value function V* and the determination of the optimal refinancing strategy consisting of an N-uple of T-valued strictly increasing stopping times.

a. **Time T value function**

At time T the debtor pays off the time T–1 standing capital plus the interest accrued between T–1 and T. Then the contract extinguishes and becomes worthless:

$$V(T, (y, x, r, n)) = 0,$$

for all admissible y, x, r, and n.

b. Time t value function

The value function at time t is to be computed by selecting the cheapest between the cost for refinancing the mortgage (CR) and the cost resulting from continuing under the standing debt rate conditions (CNR). Of course, if no refinancing opportunity is still available, the holder needs to bear the cost of continuing the contract.

$$V(t,(y,x,r,n)) = \begin{cases} \min\{CR(t,y,x,r,n), CNR(t,y,x,r,n)\}, \text{ if } n = 1,\ldots,N, \\ CNR^0(t,y,x,r), \text{ if } n = 0. \end{cases} \quad (6)$$

Here, intensive computations are required.

$$\begin{aligned} CR(t,y,x,r,n) &= CP_{[t,t+1]}(y,x)+DEC(t+1,y,x,r,n-1) \\ &= y\times c(x)\times T^*(t)+ E^{Px}(V(t,(B(t+1),R(t+1)(\bullet),x,n-1))) \end{aligned} \quad (7)$$

where $T^*(t)= 1/(T-t)$ is the complement to one of the coefficient T(t) and the symbol "•" indicates the argument with respect to which expectation is to be computed. Noting that 1) the time t+1 standing balance is equal to $B(t+1)=c(r(t))\times B(t)\times T(t)$, 2) the expected value is computed over the possible values taken by the market rate R(t+1), and 3) V factors in the product $y\times V^*$, the cost for refinancing becomes:

$$\begin{aligned} CR &= y\times c(x)\times T^*(t)+E^{Px}(V(t+1,(c(x)\times y\times T(t), R(t+1)(\bullet), x, n-1))) \\ &= y\times c(x)\times T^*(t)+c(x)\times T(t)\times y\times E^{Px}(V^*(t+1,(R(t+1)(\bullet), x, n-1))) \end{aligned}$$

Analogous computations for the cost of continuation lead to

$$\begin{aligned} CRN &= CP_{[t,t+1]}(y,r)+DEC(t+1,y,x,r,n) \\ &= y\times c(r)\times T^*(t)+E^{Px}(V(t+1,(B(t+1)(\bullet), r, n))) \\ &= y\times c(r)\times T^*(t)+c(r)\times T(t)\times y\times E^{Px}(V^*(t+1,(R(t+1)(\bullet), r, n))) \end{aligned}$$

The same cost of continuation when all options have been exercised, i.e., n=0, reads as

$$CNR^0(t,y,x,r)=CNR(t,y,x,r,0).$$

By gathering these expressions altogether and dividing by the outstanding balance y we come up to a formula for the time t unitary value function:

$$V^{*} = \begin{Bmatrix} \min\{c(x)T^{*}(t) + c(x)T(t)E^{Px}(V^{*}(t+1,(R(t+1)(\bullet),x,n-1))), \\ ,c(r)T^{*}(t) + c(r)T(t)E^{Px}(V^{*}(t+1,(R(t+1)(\bullet),r,n)))\} \quad n \geq 1 \\ c(r)T^{*}(t) + c(r)T(t)E^{Px}(V^{*}(t+1,(R(t+1)(\bullet),r,0))) \quad n = 0 \end{Bmatrix}.$$

For the purpose of determining the optimal stopping rule at time t an explicit minimization is required to decide whether to refinance or not. We finally remark that the probability under which expectations are taken is determined by (1) through the rule $P_x(\{y\}) := P(x, y)$ in all cases.

c. Time 0 value function

At time t=0 the unitary value function V* becomes independent of variables r and n. Indeed, this is the first time the debtor is allowed to refinance and all the N options are available. Note that the debt rate on [0,1] is r_0 and $R(1) \in \{r_0, r_0+\Delta s, r_0-\Delta s\}$. The value function is given by

$$V = V^{*}(0,(r_0,r_0,N)) = c(r_0) \times T^{*}(0) + c(r_0) \times T^{*}(0) \times E^{Pr_0}(V^{*}(1,(\bullet,r_0,N)))$$

This is the fair price of the mortgage contract at time 0.

IV. TRANSACTION COSTS

To provide an effective disincentive many mortgages prescribe a fee to apply upon prepayment: the mortgagor wishing to extinguish the contract is required to pay an additional fee usually ranging from 0.5 to 2 percentage points over the outstanding balance. A slight modification is required to accommodate this feature with our framework. Again, the value function factorizes as

$$V(t,(y,x,r,n)) = y \times V^{*}(t,(x,r,n)) \tag{8}$$

Since transaction costs are proportional to the outstanding balance, they can be included into the value function as a constant adding to the function CR. Let ϕ represent the outstanding balance quota that is singled out as a prepayment fee. In general, the time t value function $V^{*}=V^{*}(t,(x,r,n))$ per unit balance is computed as

$$V^{*} = \begin{Bmatrix} \min\{c(x)T^{*}(t) + c(x)T(t)E^{Px}(V^{*}(t+1,(R(t+1)(\bullet),x,n-1))) + \phi, \\ ,c(r)T^{*}(t) + c(r)T(t)E^{Px}(V^{*}(t+1,(R(t+1)(\bullet),r,n)))\} \quad n \geq 1 \\ c(r)T^{*}(t) + c(r)T(t)E^{Px}(V^{*}(t+1,(R(t+1)(\bullet),r,0))) \quad n = 0 \end{Bmatrix}$$

V. EMPIRICAL RESULTS

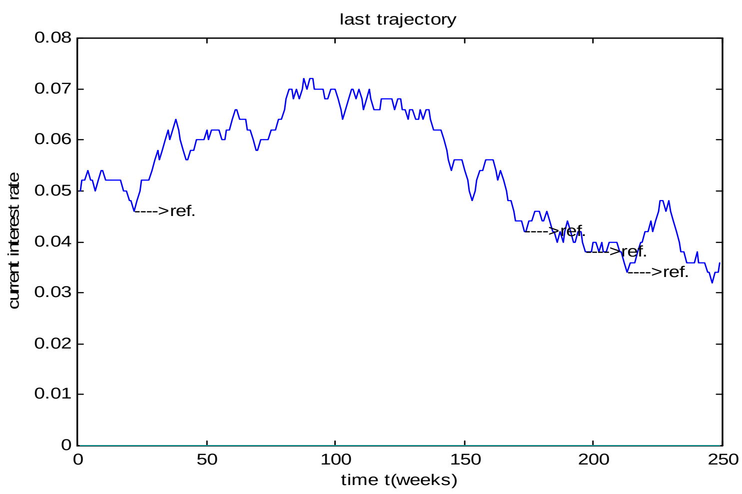
We computed the fair value of the contract $V(0,(1,r_0,N))$ corresponding to a standing rate r_0=5% an a number N=4 of refinancing opportunities. This is the minimal cost for the mortgagor, namely the one resulting from the optimal control policy. We obtain an optimal cost equal to 1.4404.

The optimal refinancing policy is expressed as a rule transforming any possible configuration of the state variable into an action consisting of either continuing or refinancing the mortgage. To have a concrete idea about the optimal policy, we plot a sample path of the market rate process and indicate the four optimal refinancing times (see Figure 1). Notice that the first refinancing option is exercised after a few weeks compared to the repayment horizon. While market rates increase for about 75 weeks, no refinancing takes place. In the following weeks, market rates steadily decrease and a new refinancing option is exercised as soon as they go below the standing debt rate. The remaining two options are exercised in a relatively narrow time window, always at points of minimum along the rate path. Naturally, this behavior is specific to the examined path and other samples may give rise to different control actions.

A clearer picture of the optimal control policy is obtained by computing the distributions of the four optimal refinancing times. Analytical expressions for these distributions are not available. However, we may compute a sample estimate of their shape by simulating a large number of trajectories, then storing the corresponding optimal stopping times, and finally computing the relative frequency histogram of the four refinancing times. Figure 2 shows a sample probability distribution for the first two optimal refinancing times as obtained through 10,000 sample paths of the market rate process. The first option tends to be exercised within the first two weeks, whereas the second stopping time seems to span a wider time period. It is interesting to note that a refinancing option is never exercised during the first period. The first refinancing time displays a spiky distribution with a maximum attained at time t=2. It seems that if the market rate decreases during the first period, the high proximity to the initial period tends to offset the advantage stemming from refinancing. On the contrary, if the market rate decreases in the second period, this effect reverses and the mortgage is refinanced.

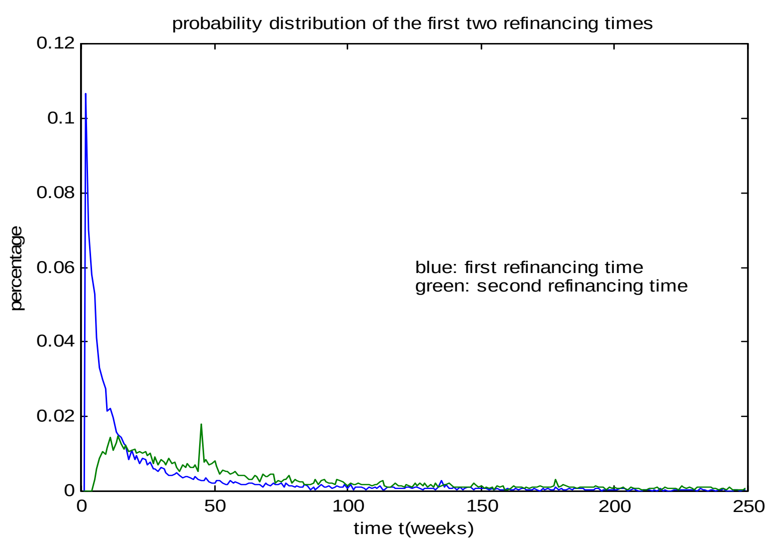
Figure 3 shows the last two refinancing times. These times cluster on the farthest end of the horizon spectrum. However, they also display a marked spiky behavior. We also computed the empirical relative frequency of the first refinancing time over 10,000 and 100,000 sample paths. Figure 4 displays the two distributions obtained by independent path generations. The two graphs exhibit similar properties, showing that convergence to the exact distribution of the first refinancing time is fairly quick. In particular, the large spike occurring in about 50 weeks seems to be an intrinsic feature of the control policy.

Figure 1



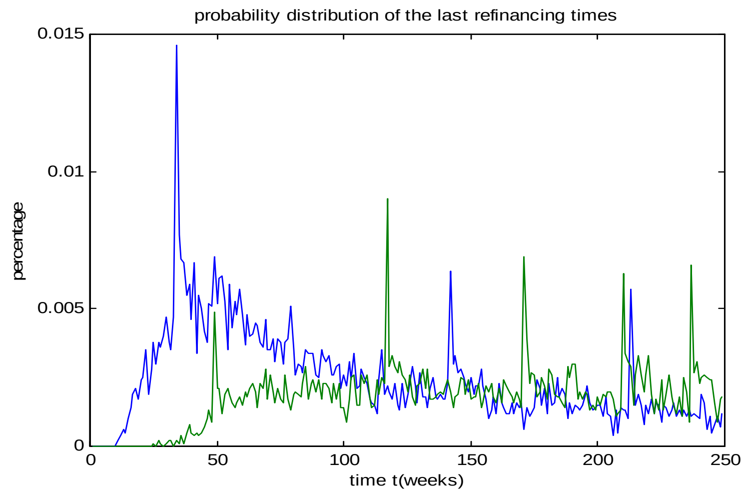
Note: Sample trajectory of the market rate of interest. Expression "→ ref " indicates the four optimal refinancing times.

Figure 2



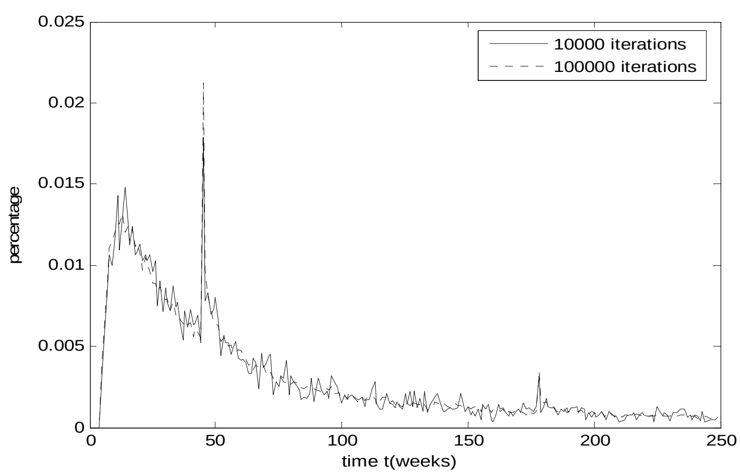
Notes: Sample probability distribution of the first two refinancing times over n = 10,000 sample paths.

Figure 3



Notes: Sample probability distribution of the last two refinancing times over n = 10,000 sample paths.

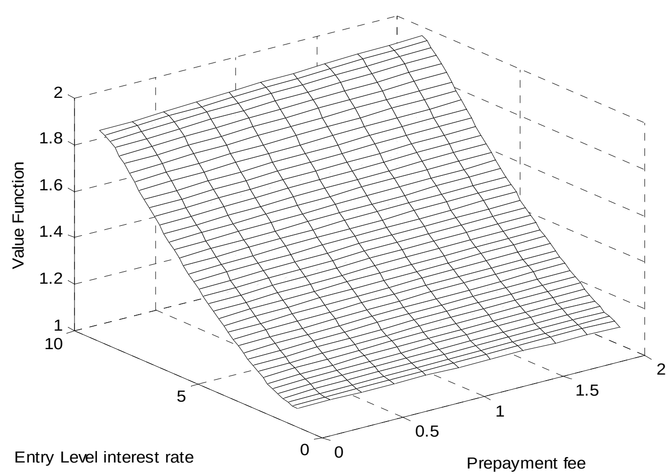
Figure 4



Notes: First optimal refinancing time sample distributions computed over alternative sample sizes. Case 1: (plain line) sample size = 10,000. Case 2: (dotted line) sample size = 100,000.

Our final experiment checks for the behavior of the mortgage value as a function of the entry level mortgage rate r_0 and an additional fee is applied to any refinancing decision. Interest rates vary from 1 to 9 percentage points. Prepayment fees range from 0 to 2 percentage points over the standing capital. The contract is supposed to expire in 180 months and 4 refinancing opportunities are allotted at the outset. Figure 5 reports a plot of the corresponding two-dimensional surface. It is clear that higher entry rates or larger fees have a negative effect on the minimal cost attainable by the mortgage holder. This is reflected by an increase in the time 0 value function as shown by the graph.

Figure 5



Notes: Mortgage value under optimal refinancing policy across varying entry-level mortgage rates and prepayment fees.

VI. CONCLUSION

We have introduced a model for the endogenous determination of the optimal refinancing policy for mortgage loans under constrained refinancing opportunities. Dynamic programming allowed us to explicitly compute the strategy in terms of three state variables: the standing debt rate, the current market rate, and the residual number of refinancing options. The model can easily be extended to include transaction costs incurring upon a prepayment. We have numerically computed the distribution of multiple exercise policies under alternative scenarios and have shown the joint impact of entry-level interest rates and prepayment fees on the value of the mortgage.

ENDNOTES

1. Specifically, the repayment process ($P^{\alpha}(t), 0 \leq t < T$) is defined on the probability space ($S^T, B(S^T), P_{R(0)}$) and the expectation is performed under the probability measure $P_{R(0)}$ induced on the space of paths by the transition probabilities (1) and the initial condition R(0).

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