INTERNATIONAL JOURNAL OF BUSINESS, 17(2), 2012 ISSN: 1083-4346

A Few Insights into Cliquet Options

Tristan Guillaume
Université de Cergy-Pontoise, Laboratoire Thema,
33 boulevard du port, 95011 Cergy-Pontoise Cedex, France
Tristan.Guillaume@u-cergy.fr

ABSTRACT

This paper deals with a subset of lookback options known as cliquet options. The latter lock in the best underlying asset price over a number of prespecified dates during the option life. The specific uses of these constracts are analyzed, as well as two different hedging techniques. Closed form valuation formulae are provided under standard assumptions. They are easy to implement, very efficient and accurate compared to Monte Carlo simulation approximations.

JEL Classifications: *G13, G12*

Keywords: *Cliquet option; lookback option; option valuation; option hedging; numerical dimension*

I. INTRODUCTION

The term “cliquet option” is ambiguous. It may refer either to a subset of the lookback options characterized by a small number of fixing dates that are not necessarily uniformly spaced, or to portfolios of forward start options. The latter are also known as reset strike options or ratchet options, to avoid confusion. This paper deals with the former kind of contract, that does not raise the same pricing and hedging issues as reset strike options. Cliquet options, in this sense, are heavily traded in the markets, especially as building blocks for many structured products. In particular, a lot of capital-guaranteed investments provide a participation in an index based on a percentage of the best recorded index value over a number of contractually prespecified dates. There are also a lot of equity-linked bonds whose rate of interest is based on a similar formula.

Generally speaking, cliquet options are appealing to investors because they inherit, at least partially, the very attractive payoff of lookback options, while rendering it both more affordable and more flexible, thanks to the decrease in the updating frequency of the running extremum of the underlying asset price, as well as to the possible partial and non uniform spanning of the option life. They also have many advantages compared to other options that allow their holders to « lock in » positive intrinsic values prior to expiry, such as ladder or shout options. In particular, unlike ladder options, they do not cap the greatest possible intrinsic value from the start, and they do not require their holders to have precise anticipations on a set of target price increases in the underlying asset.

However, cliquet options raise a number of pricing and hedging issues. Dynamic hedging is costly because of sharp delta increases during the option life and oscillations of gamma between fixing dates. Pure static hedging is not possible. A semi-static strategy can replicate the option payoff but it is difficult to translate it into a reliable fair price for the option because it is heavily exposed to volatility risk. Finally, it is not easy to come up with a convenient analytical solution of the pricing problem because of quickly increasing dimension.

This paper attempts to address these various issues. In Section II, the specific properties of cliquet options are highlighted, in comparison with alternative contracts traded in the markets such as lookback, ladder and shout options. Then, the difficulties associated with dynamic hedging are briefly analyzed, leading to the presentation of a semi-static rollover strategy to replicate the cliquet option payoff. In Section III, closed form valuation formulae are provided, first in dimension 3 and 4, then for any number of fixing dates. The numerical implementation of these formulae is dealt with in the remainder of Section III, while Section IV provides a proof of some of the analytical results in Section III.

II. SPECIFIC PROPERTIES OF CLIQUET OPTIONS

The idea of locking-in positive intrinsic values prior to the option’s expiry is embedded in a number of European-type financial options. The type of contract that provides this feature to its fullest extent is the lookback option, whose payoff at expiry T writes (for a call with fixed strike):

$$\left(\sup_{0\le t\le T} S(t) - K\right)^{+} \qquad (2.1)$$

where S(t), $t \geq 0$ is the underlying asset and K is the strike price.

Instead of taking into account only the terminal asset value, all the prices quoted from time 0 to time T are factored in when computing the option payoff. The running maximum of S(t) is continuously and automatically updated. Taking K=S(0), that is, defining the strike price as the value of the underlying at the contract's inception, this contract guarantees its holder that he or she will sell the underlying at the highest in the interval [0,T]. Buying at the lowest and selling at the highest, these are the ultimate goals of equity investors. The difficult problem of market timing, whether it be market entry or market exit, is thus automatically solved in an optimal manner by such contracts, that were originally valued by Conze and Viswanathan (1991).

The hi-lo option takes this idea even further, as it automatically captures the greatest variability of the underlying asset over the option life, yielding at expiry the following difference:

$$\left(\sup_{0\le t\le T} S(t) - \inf_{0\le t\le T} S(t)\right) \qquad (2.2)$$

The main drawback of these two types of contracts is quite obvious: they are very expensive, so that option holders are rather unlikely to reach the breakeven point and thus receive a positive return from their investment. One simple way to lower the cost of lookback options consists in defining a discrete updating frequency of the running maximum or minimum. By the way, there can never be truly continuous updating in practice, even for the most liquid traded stocks, because of the minimum time gap between two quotations in the market, which can be very short but never arbitrarily short. It is clear that decreasing the updating frequency to, say, one day (closing market price) or, better, one week, will result in a lower expected value of the option and hence a lower premium. Such contracts are called discretely monitored lookback options. They preserve the global lookback property since there are still many equally spaced fixing dates, but not to its full extent.

Cliquet options are essentially a form of discretely monitored lookback options, albeit with three differences in practice:

- there are usually few fixing dates
- the lookback period may not span the entire option life
- the fixing dates are not always equally spaced

Denoting by $\{t_1,t_2,\ldots,t_n\}$ the set of the n times at which positive intrinsic option values can be locked in - these times being called the fixing dates, with $t(0)=0<t_1<t_2<\ldots<t_n=T$, one can write the payoff of a cliquet call as follows :

$$(\max(S(t_1), S(t_2), \ldots, S(t_n)) - K)^{+} \qquad (2.3)$$

It must be pointed out that, in the markets, the term "cliquet", when applied to a call option, is sometimes used interchangeably to refer to a ratchet option, whereby the strike price resets at each fixing date to the prevailing stock price while at the same time locking in the performance of the previous period, thus yielding at expiry the following payoff:

$$\sum_{i=0}^{n} \max\left[S(t_{i+1}) - S(t_i), 0\right] \quad (2.4)$$

Compared to payoff (2.3), payoff (2.4) has no fixed strike and adds every positive contribution up until expiry. Ratchet options are simply a portfolio of forward starting options, hence they admit perfect static replication, unlike cliquet options (Buetow, 1999; Matosek, 2008).

There are no restrictions on the location of the fixing dates over [0,T] in a cliquet option contract. In particular, they may all be concentrated in a specific subinterval of [0,T] and leave large parts of [0,T] non monitored. This flexibility is valuable to investors because it enables them to tailor the option contract to their particular anticipations or hedging requirements, while it substantially reduces the option premium by waiving the lookback property when it is not needed.

The specificity of cliquet options is better grasped through a short analysis of their differences with alternative contracts that also embed a lock-in mechanism. For instance, ladder options are contracts whereby the set of the underlying asset price levels that can be locked in before expiry is prespecified at time zero - they are called the “rungs”. Denoting by L the highest level or rung reached over the option life, a ladder call then provides its holder with the following payoff:

$$\max\left[S(T) - K, L - K, 0\right] \quad (2.5)$$

Ladder options can be valued as combinations of barrier options and vanilla options (De Weert, 2008). The rungs may be attained at any time prior to T, but all prices other than the rungs and the terminal value of S will not be taken into account when computing the option payoff. In other words, the difference with a cliquet option is two-fold: there are no fixing dates and the maximum level that can be locked in is bounded. One advantage, as far as risk-averse investors are concerned, is that if they set the first rung of the ladder close enough to the spot price at the contract’s inception, they are very likely to receive a minimum positive payoff, due to the high probability that the first rung will be attained. However, by definition, this minimum positive payoff is low. You need to add more and more rungs if you want the expected payoff to grow. But then, the premium of the ladder option gradually tends to that of a lookback option. If you think that the actual realized volatility of the underlying will be higher than the implied volatility priced by the market, then you might want to elevate the first rung in your ladder option to lower the premium but it is clearly risky to do so.

In contrast to a ladder option, a cliquet option does not require its holder to have precise anticipations on a set of target price increases in the underlying asset, which is attractive in uncertain markets or when you do not have enough expertise. Also, cliquet options do not involve capping the greatest possible intrinsic value from the start. Such a capping mechanism obviously does not allow to fully exploit market movements. Moreover, unlike ladder-type contracts, cliquet options enable their holders to take advantage of specific time-related views or constraints. For instance, if there is a strong case that the underlying asset price will probably reach its maximum during a given subinterval of [0,T], because of scheduled events such a public announcement liable to provide impetus to the stock price of a company or elections likely to have a major impact on a currency, then the investor had better locate most of the fixing dates in this specific time area. Indeed, the shorter the lock-in period, the larger the drop in the option premium. Ultimately, you might even be able to generate a very high payoff with a single fixing date prior to expiry, which makes the option very affordable.

If you are uncertain about both the magnitude and the timing of the underlying asset price increases, an alternative strategy is to turn to a shout option. The latter allows you to lock in the intrinsic value of the option at any time prior to expiry – an action referred to as a « shout », thus providing the following payoff at maturity (for a call):

$$\max\left[S(T)-K, S\left(t^*\right)-K, 0\right] \tag{2.6}$$

where t^* is the time chosen to "shout".

A shout option is valued in much the same way as an American option (Windcliff et al., 2001). This contract is well suited to investors who wish to play a more active role during the option life and who are not satisfied with the fact that a contract would prespecify either the possible lock-in price levels (ladder option) or the times at which the underlying asset price can be locked in (cliquet option). Compared to a ladder option, one advantage is that you are not stuck at a rung: if the underlying asset price reaches the otherwise highest prespecified rung and keeps on rising, you can wait before you shout in order to lock in even higher levels. Also, you can lock in any positive intrinsic value before expiry, including all those below the first rung or between two different rungs of a ladder option. Relative to a cliquet option, the holder of a shout option is free to wait and see how the market moves before deciding when to lock in an underlying asset price. On the downside, this generates uncertainty as you never know when it is optimal to shout, so that you are faced with the possibility of subsequent regrets, whereas a cliquet option provides as many possibilities of locking in asset price levels as there are fixing dates. Nothing theoretically rules out allowing for several "shouts" during the option life but then the option quickly becomes more expensive than a cliquet option. Also, shout options require steady monitoring and a lot of investors do not want to dedicate too much of their time to manage their option position. Lastly, from the point of view of derivatives sellers, shout options are difficult to hedge because of the unpredictability of the exercice time. For all these reasons, cliquet options are more popular than shout options in the markets.

As far as hedging is concerned, there are differences between the dynamics of the delta of a cliquet call and those of a vanilla call. In general, unless the option is then deeply out-of-the-money, the delta of a cliquet call will significantly increase in the vicinity of a fixing date as a consequence of the lock-in mechanism. Before the first fixing date t_1, this phenomenon is observed especially when there is little time left before t_1 and when the underlying asset is then trading near the strike. If it is then trading far below the strike, though, the delta of the cliquet call will be quite similar to that of a vanilla call.

If $S(t_1)>K$, the holder of the cliquet call option is sure to receive at least $S(t_1)>K$ at expiry. This is why the value of a cliquet call is less sensitive than that of a vanilla call to a decline in the value of S below the level $S(t_1)$ after time t_1. It will remain sensitive to such in a decline in the value of S, still, as the latter has a negative impact on the probability that S will reach levels above $S(t_1)$ at fixing dates subsequent to t_1. But at least, unlike a vanilla call, a decline in the value of S below the level $S(t_1)$ after time t_1 has no negative impact on the probability that the option will end in-the-money at expiry.

In the vicinity of the second fixing date, t_2, the magnitude of the delta increase will depend on the current and past levels of the underlying. If the difference $S(t_1)$-K is positive and S is trading near the level $S(t_1)$ little before t_2, then the delta increase is sharp. If S rises above the level $S(t_1)$ and the time is close to t_2, then the delta of the cliquet call tends to 1 and the gamma drops to zero. The more volatility around the level $S(t_1)$ between t_1 and t_2, the more delta oscillates, with gamma reaching a local maximum at the level $S(t_1)$ just before t_2.

The same analysis carries over to any two consecutive fixing dates. Overall, there may be several areas of sharp delta fluctuations during the option life. These fluctuations will be particularly pronounced when the time is near the last fixing date and the underlying asset is trading near the latest locked-in maximum value. Thus, a delta hedging strategy may turn out to be very costly on a long maturity option, as high transaction costs are incurred due to the frequent rebalancing of the hedging portfolio. This issue is compounded when the option is written on an asset with limited liquidity or jumpy returns, as explained by Petrelli et al. (2008).

This is why, in practice, the sale of a cliquet option may be hedged by implementing a semi-static rollover strategy. Assuming there are three only fixing dates, t_1, t_2, t_3 and the option expiry is t_3, the semi-static hedge of the sale of a cliquet call would proceed as follows:

- at time zero, buy a vanilla call struck at K and expiring at t_1.
- at time t_1 if $S(t_1)>K$ then buy a new call struck at $S(t_1)$ and expiring at t_2; if $S(t_1)<K$ then buy a new call struck at K and expiring at t_2.
- at time t_2, if $S(t_2)>K$ then buy a new call struck at $S(t_2)$ and expiring at t_3; if $S(t_2)<K$ and $S(t_1)<K$ then buy a new call struck at K and expiring at t_3; if $S(t_2)<K$ and $S(t_1)>K$ then buy a new call struck at $S(t_1)$ and expiring at t_3.

The same procedure easily extends to a higher number of fixing dates. However, it does not work without costs. First, the ability to implement the hedge depends on the availability of the required options in the market, which may be a problem if the underlying is not liquid enough or if the maturities are non-standard. Even if the required options are traded, there are vanilla option bid-ask spreads that need to be factored in.

Besides, it is not easy to value the cost of this semi-static rollover strategy at the time the cliquet call is sold. Indeed, it involves purchasing options in the future whose strikes are not known at time zero and depend on future values of S. Numerical methods must be used to approximate the value of these future contracts. Whatever numerical methods are used, the skew must also be taken into account to obtain a reliable price. When $S(t_1)<K$ and $S(t_2)<K$, both vanilla options bought at times t_1 and t_2 are struck at K. To value the former at time zero, one should look at the $[t_1, t_2]$ forward implied volatility point on the volatility surface. It is important to bear in mind that the

associated skew is the [$t_{1,}$ t_2] one, which is typically steeper than the [$t_{0,}$ t_2] one. The farther t_1 is from t_0, the higher the chances that the [$t_{1,}$ t_2] option will be priced at time zero using an inappropriate skew, because there is uncertainty about the level K/S at time t_1 and about the variation in the slope of the skew curve between t_0 and t_1. These problems are compounded when it comes to the valuation of the [$t_{1,}$ t_2] option in the semi-static hedge. Next, when $S(t_1)>K$ and $S(t_2)>K$, both vanilla options bought at times t_1 and t_2 are struck at a level above K but they are at-the-money at the time of purchase. Thus, the associated skew should be considered as an at-the-money forward skew, while traders blindly pricing these options off the spot volatility surface would use a volatility estimate that would be too low. Finally, when $S(t_1)<K$ and $S(t_2)>K$, and when $S(t_1)>K$ and $S(t_2)<K$, the variation in implied volatility caused by the skew varies according to the relative levels of $S(t_1)$ and $S(t_2)$ as well as on the relative distance of t_1 and t_2 to the option expiry; as the skew is steeper for short term maturities and the skew curve tends to flatten or even have an element of a smile shape for very high strikes, the net effect on implied volatility is likely to be strongly upward when $S(t_1)<K$ and $S(t_2)>K$. Thus, the skew exposure is quite difficult to manage; for more material on this important practical issue, the reader is referred to Gatheral (2006), who shows that pricing this kind of structure with a local volatility model tends to underestimate the real value, relative to a stochastic volatility model, as it produces forward skews that are too flat.

There are also issues associated with the risk management of sensitivities other than delta and gamma, such as the vega of a cliquet option. The latter is typically higher than the vega of a vanilla option. Indeed, the larger the volatility of the underlying, the greater the probability to reach a new extremum on each fixing date. Moreover, as each new extremum is locked in, the option value is not exposed to the higher chance of adverse price movements. Another difference with the vega of a vanilla option is that the vega of a cliquet option is at its peak when the underlying is trading near the latest extremum that was locked in at one of the previous fixing dates, not when it is trading near the strike price. However, one must bear in mind that the vega of a cliquet option may be a misleading measure of volatility risk because this is a contract whose gamma changes sign and, as a result, vega may be small at precisely those places where sensitivity to actual volatility is very large. This issue is discussed by Wilmott (2002), who analyzes the implications of the fact that the point at which gamma changes sign depends on the relative move in S from one fixing to the next.

III. VALUATION FORMULAE AND NUMERICAL IMPLEMENTATION

As pointed out in the previous section, cliquet options may be hedged by a semi-static rollover strategy, but the latter is difficult to value at the time the option is sold. Therefore, it would be quite convenient to be able to refer to an easily computed no-arbitrage price formula, at least as an analytical benchmark to test more general numerical schemes. Unfortunately, cliquet options are not easy to price by analytical methods because of the rapidly increasing dimension of the valuation problem. To put it simply, the more fixing dates, the higher the dimension, as measured by the required order of numerical integration. The main problem associated with increasing dimension is that analytical results rely on functions that are defined as multiple integrals that may not be easily computed or that may not even be known. As a consequence of this dimension issue, the only closed form solution that has been published so far is a formula for a cliquet option with a single fixing date prior to expiry (Gray and Whaley, 1999) in a standard Black-Scholes framework. Other contributions focus on numerical methods, via either Monte Carlo simulation or a partial differential equation solver (Windcliff et al., 2006). It must be pointed out that the classical continuity correction of Broadie et al. (1999) used to provide an analytical approximation of the price of discretely monitored lookback options is unfortunately quite inaccurate when there are only few fixing dates, especially when the latter are not uniformly spaced over the option life, and it is therefore not suited to cliquet options.

The following Proposition 1 provides a closed form solution for the no-arbitrage value, V, of a cliquet option with expiry t_3, three fixing dates, t_1, t_2, and t_3, assuming that the underlying asset S is driven by a geometric Brownian motion with constant volatility σ, payout rate δ and risk-free interest rate r.

The following expressions will be used, with $(i,j,m,n) \in N^4, i<j, m<n$, where λ=1 if the option is a call, λ=-1if the option is a put.

$$k = \ln(K/S(t_0))$$
$$\mu_{(i)} = (r - \sigma^2/2)t_i, \quad \bar{\mu}_{(i)} = (r+\sigma^2/2)t_i$$
$$\mu_{(i,j)} = (r - \sigma^2/2)(t_j - t_i), \quad \bar{\mu}_{(i,j)} = (r+\sigma^2/2)(t_j - t_i)$$
$$\sigma_{(i)} = \sigma\sqrt{t_i}, \quad \sigma_{(i,j)} = \sigma\sqrt{t_j - t_i}$$
$$b(i,j) = \sqrt{\frac{t_i}{t_j}}, \ \rho(i,j) = \sqrt{1-\frac{t_i}{t_j}}, \quad \rho(i,j;m,n) = \sqrt{\frac{t_j - t_i}{t_n - t_m}}$$

Also, the function $\phi_n(a_1, a_2, \ldots, a_n; \theta_1, \theta_2, \ldots, \theta_{n-1}), \ \ a_i \in R, \theta_i \in]-1,1[, i \in N$, is defined as follows: for n=1, ϕ_n is the univariate standard gaussian cumulative distribution function ; for n=2, ϕ_n is the bivariate standard gaussian cumulative distribution function ; for n>2, ϕ_n is given by :

$$\Phi_n = \int_{x_1=-\infty}^{a_1} \int_{x_2=-\infty}^{a_2} \ldots \int_{x_n=-\infty}^{a_n} \frac{\exp\left(-\frac{x_1^2}{2} - \sum_{i=1}^{n-1} \frac{(x_{i+1} - \theta_i x_i)^2}{2\left(1-\theta_i^2\right)}\right)}{(2\pi)^{n/2} \prod_{i=1}^{n-1} \sqrt{\left(1-\theta_i^2\right)}} dx_1 dx_2 \ldots dx_n \tag{3.1}$$

Proposition 1 can now be stated.

A. Proposition 1

$$V(S(t_0), \sigma, \delta, r, K, \{t_1, t_2, t_3\}) \tag{3.2}$$

$$
\begin{aligned}
&= \lambda \times \exp(-r(t_3 - t_1) - \delta t_1) \times S(t_0) \times \Phi_1\left[\lambda \times \frac{-k + \bar{\mu}_{(1)}}{\sigma_{(1)}}\right] \\
&\quad \times \Phi_2\left[\lambda \times \frac{-\mu_{(1,2)}}{\sigma_{(1,2)}}, \lambda \times \frac{-\mu_{(1,3)}}{\sigma_{(1,3)}}; \rho_{(1,2;1,3)}\right] \\
&\quad - \lambda \times \exp(-rt_3) \times K \times \Phi_1\left[\lambda \times \frac{-k + \mu_{(1)}}{\sigma_{(1)}}\right] \times \Phi_2\left[\lambda \times \frac{-\mu_{(1,2)}}{\sigma_{(1,2)}}, \lambda \times \frac{-\mu_{(1,3)}}{\sigma_{(1,3)}}; \rho_{(1,2;1,3)}\right] \\
&\quad + \lambda \times \exp(-r(t_3 - t_2) - \delta t_2) \times S(t_0) \times \Phi_1\left[\lambda \times \frac{-\mu_{(2,3)}}{\sigma_{(2,3)}}\right] \\
&\quad \times \left(\Phi_2\left[\lambda \times \frac{k - \bar{\mu}_{(1)}}{\sigma_{(1)}}, \lambda \times \frac{-k + \bar{\mu}_{(2)}}{\sigma_{(2)}}; -\beta_{(1,2)}\right] + \Phi_1\left[\lambda \times \frac{-k + \bar{\mu}_{(1)}}{\sigma_{(1)}}\right] \times \Phi_1\left[\lambda \times \frac{\bar{\mu}_{(1,2)}}{\sigma_{(1,2)}}\right]\right) \\
&\quad - \lambda \times \exp(-rt_3) \times K \times \Phi_1\left[\lambda \times \frac{-\mu_{(2,3)}}{\sigma_{(2,3)}}\right] \\
&\quad \times \left(\Phi_2\left[\lambda \times \frac{k - \mu_{(1)}}{\sigma_{(1)}}, \lambda \times \frac{-k + \mu_{(2)}}{\sigma_{(2)}}; -\beta_{(1,2)}\right] + \Phi_1\left[\lambda \times \frac{-k + \mu_{(1)}}{\sigma_{(1)}}\right] \times \Phi_1\left[\lambda \times \frac{\mu_{(1,2)}}{\sigma_{(1,2)}}\right]\right) \\
&\quad + \lambda \times \exp(-\delta t_3) \times S(t_0) \times \Phi_3\left[\lambda \times \frac{-k + \bar{\mu}_{(3)}}{\sigma_{(3)}}, \lambda \times \frac{\bar{\mu}_{(1,3)}}{\sigma_{(1,3)}}, \lambda \times \frac{\bar{\mu}_{(2,3)}}{\sigma_{(2,3)}}; \rho_{(1;3)}, \rho_{(2,3;1,3)}\right] \\
&\quad - \lambda \times \exp(-rt_3) \times K \times \Phi_3\left[\lambda \times \frac{-k + \mu_{(3)}}{\sigma_{(3)}}, \lambda \times \frac{\mu_{(1,3)}}{\sigma_{(1,3)}}, \lambda \times \frac{\mu_{(2,3)}}{\sigma_{(2,3)}}; \rho_{(1;3)}, \rho_{(2,3;1,3)}\right]
\end{aligned}
$$

A proof of Proposition 1 is provided in the Appendix.

Next, Proposition 2 gives a closed form solution for the no-arbitrage value, V, of a cliquet option with expiry t_4, and four fixing dates, t_1, t_2, t_3 and t_4, under the same assumptions as in Proposition 1.

B. Proposition 2

$$
\begin{aligned}
&V(S(t_0), \sigma, \delta, r, K, \{t_1, t_2, t_3, t_4\}) \qquad (3.3) \\
&= \lambda \times \exp(-r(t_4 - t_1) - \delta t_1) \times S(t_0) \\
&\quad \times \left(\Phi_1\left[\lambda \times \frac{-k + \bar{\mu}_{(1)}}{\sigma_{(1)}}\right] \times \Phi_3\left[\lambda \times \frac{-\mu_{(1,2)}}{\sigma_{(1,2)}}, \lambda \times \frac{-\mu_{(1,3)}}{\sigma_{(1,3)}}, \lambda \times \frac{-\mu_{(1,4)}}{\sigma_{(1,4)}}; \rho_{(1,2;1,3)}, \rho_{(1,3;1,4)}\right]\right) \\
&\quad - \lambda \times \exp(-rt_4) \times K \\
&\quad \times \left(\Phi_1\left[\lambda \times \frac{-k + \mu_{(1)}}{\sigma_{(1)}}\right] \times \Phi_3\left[\lambda \times \frac{-\mu_{(1,2)}}{\sigma_{(1,2)}}, \lambda \times \frac{-\mu_{(1,3)}}{\sigma_{(1,3)}}, \lambda \times \frac{-\mu_{(1,4)}}{\sigma_{(1,4)}}; \rho_{(1,2;1,3)}, \rho_{(1,3;1,4)}\right]\right) \\
&\quad + \lambda \times \exp(-r(t_4 - t_2) - \delta t_2) \times S(t_0)
\end{aligned}
$$

$$\times\left(\Phi_2\left[\lambda\times\frac{k-\bar{\mu}_{(1)}}{\sigma_{(1)}},\lambda\times\frac{-k+\bar{\mu}_{(2)}}{\sigma_{(2)}};-\beta_{(1,2)}\right]+\Phi_1\left[\lambda\times\frac{-k+\bar{\mu}_{(1)}}{\sigma_{(1)}}\right]\times\Phi_1\left[\lambda\times\frac{\bar{\mu}_{(1,2)}}{\sigma_{(1,2)}}\right]\right)$$

$$\times\Phi_2\left[\lambda\times\frac{-\mu_{(2,3)}}{\sigma_{(2,3)}},\lambda\times\frac{-\mu_{(2,4)}}{\sigma_{(2,4)}};\rho_{(2,3;2,4)}\right]$$

$$-\lambda\times\exp(-rt_4)\times K$$

$$\times\left(\Phi_2\left[\lambda\times\frac{k-\mu_{(1)}}{\sigma_{(1)}},\lambda\times\frac{-k+\mu_{(2)}}{\sigma_{(2)}};-\beta_{(1,2)}\right]+\Phi_1\left[\lambda\times\frac{-k+\mu_{(1)}}{\sigma_{(1)}}\right]\times\Phi_1\left[\lambda\times\frac{\mu_{(1,2)}}{\sigma_{(1,2)}}\right]\right)$$

$$\times\Phi_2\left[\lambda\times\frac{-\mu_{(2,3)}}{\sigma_{(2,3)}},\lambda\times\frac{-\mu_{(2,4)}}{\sigma_{(2,4)}};\rho_{(2,3;2,4)}\right]$$

$$+\lambda\times\exp(-r(t_4-t_3)-\delta t_3)\times S(t_0)$$

$$\times\Phi_3\left[\lambda\times\frac{-k+\bar{\mu}_{(3)}}{\sigma_{(3)}},\lambda\times\frac{\bar{\mu}_{(1,3)}}{\sigma_{(1,3)}},\lambda\times\frac{\bar{\mu}_{(2,3)}}{\sigma_{(2,3)}};\rho_{(1;3)},\rho_{(2,3;1,3)}\right]\times\Phi_1\left[\lambda\times\frac{-\mu_{(3,4)}}{\sigma_{(3,4)}}\right]$$

$$-\lambda\times\exp(-rt_4)\times K$$

$$\times\left(\Phi_3\left[\lambda\times\frac{-k+\mu_{(3)}}{\sigma_{(3)}},\lambda\times\frac{\mu_{(1,3)}}{\sigma_{(1,3)}},\lambda\times\frac{\mu_{(2,3)}}{\sigma_{(2,3)}};\rho_{(1,3)},\rho_{(2,3;1,3)}\right]\times\Phi_1\left[\lambda\times\frac{-\mu_{(3,4)}}{\sigma_{(3,4)}}\right]\right)$$

$$+\lambda\times\exp(\delta t_4)\times S(t_0)$$

$$\times\Phi_4\left[\lambda\times\frac{-k+\bar{\mu}_{(4)}}{\sigma_{(4)}},\lambda\times\frac{\bar{\mu}_{(1,4)}}{\sigma_{(1,4)}},\lambda\times\frac{\bar{\mu}_{(2,4)}}{\sigma_{(2,4)}},\lambda\times\frac{\bar{\mu}_{(3,4)}}{\sigma_{(3,4)}};\rho_{(1;4)},\rho_{(2,4;1,4)},\rho_{(3,4;2,4)}\right]$$

$$-\lambda\times\exp(-rt_4)\times K$$

$$\times\Phi_4\left[\lambda\times\frac{-k+\mu_{(4)}}{\sigma_{(4)}},\lambda\times\frac{\mu_{(1,4)}}{\sigma_{(1,4)}},\lambda\times\frac{\mu_{(2,4)}}{\sigma_{(2,4)}},\lambda\times\frac{\mu_{(3,4)}}{\sigma_{(3,4)}};\rho_{(1;4)},\rho_{(2,4;1,4)},\rho_{(3,4;2,4)}\right]$$

Proof of Proposition 2 is similar to that of Proposition 1 and is therefore omitted.

Finally, for a number of fixing dates greater than 4, the previous results can be generalized to produce the more compact Proposition 3.

C. Proposition 3

$$V(S(t_0),\sigma,\delta,r,K,\{t_1,t_2,...,t_n\}) \tag{3.4}$$

$$\lambda\times\left[\begin{array}{l}S(t_0)\times\left(\sum_{i=1}^{n-1}\exp(-r(t_n-t_i)-\delta t_i)\times\bar{H}_{[i]}\times Z_{[i]}+\exp(-\delta t_n)\bar{H}_{[n]}\right)\\ -\lambda\times K\times\exp(-rt_n)\times\left(\sum_{i=1}^{n-1}H_{[i]}\times Z_{[i]}+H_{[n]}\right)\end{array}\right]$$

where

$$H_{[i]} = \Phi_i\left[\begin{array}{l}\lambda \times \frac{-k+\mu_{(i)}}{\sigma_{(i)}}, \lambda \times \frac{\mu_{(1,i)}}{\sigma_{(1,i)}}, \lambda \times \frac{\mu_{(2,i)}}{\sigma_{(2,i)}}, \ldots, \lambda \times \frac{\mu_{(i-1,i)}}{\sigma_{(i-1,i)}}; \\ \rho_{(1;i)}, \rho_{(2,i;1,i)}, \rho_{(3,i;2,i)}, \ldots, \rho_{(i-1,i;i-2,i)}\end{array}\right]$$

$$\bar{H}_{[i]} = \Phi_i\left[\begin{array}{l}\lambda \times \frac{-k+\bar{\mu}_{(i)}}{\sigma_{(i)}}, \lambda \times \frac{\bar{\mu}_{(1,i)}}{\sigma_{(1,i)}}, \lambda \times \frac{\bar{\mu}_{(2,i)}}{\sigma_{(2,i)}}, \ldots, \lambda \times \frac{\bar{\mu}_{(i-1,i)}}{\sigma_{(i-1,i)}}; \\ \rho_{(1;i)}, \rho_{(2,i;1,i)}, \rho_{(3,i;2,i)}, \ldots, \rho_{(i-1,i;i-2,i)}\end{array}\right]$$

$$Z_{[i]} = \Phi_{n-i}\left[\begin{array}{l}\lambda \times \frac{-\mu_{(i,i+1)}}{\sigma_{(i,i+1)}}, \lambda \times \frac{-\mu_{(i,i+2)}}{\sigma_{(i,i+2)}}, \ldots, \lambda \times \frac{-\mu_{(i,n)}}{\sigma_{(i,n)}}; \\ \rho_{(i,i+1;i,i+2)}, \rho_{(i,i+2;i,i+3)}, \ldots, \rho_{(i,n-1;i,n)}\end{array}\right]$$

Although Proposition 3 nests Proposition 1 and Proposition 2, its terse formulation may render it relatively unclear and ambiguous without the help of the fully expanded Proposition 1 and Proposition 2, and thus it was useful to state the latter. Moreover, Proposition 2 and Proposition 3 include dimension reductions that do not appear in the more compact Proposition 3, and they are therefore more efficient to use when pricing cliquet options with three or four fixing dates.

From a numerical point of view, the valuation formulae presented in this section raise the question of the computation of the function Φ. As the number of fixing dates increases, so does the dimension of numerical integration. Fortunately, unlike lookback options, most cliquet options traded in the markets have few fixing dates, as mentioned earlier. Moreover, the dimension of numerical integration can be reduced by using the following identities:

- When there are three fixing dates, the actual numerical dimension of the function Φ_3 can be brought down from 3 to 1 by using :

$$\Phi_3[b_1, b_2, b_3; \theta_1, \theta_2] = \int_{x_2=-\infty}^{b_2} \frac{\exp\left(-x_2^2/2\right)}{\sqrt{2\pi}} N\left[\frac{b_1 - \theta_1 x_2}{\sqrt{1-\theta_1^2}}\right] N\left[\frac{b_3 - \theta_2 x_2}{\sqrt{1-\theta_2^2}}\right] dx_2 \quad (3.5)$$

- When there are four fixing dates, the actual numerical dimension of the function Φ_4 can be reduced from 4 to 2 by using :

$$\Phi_4[b_1, b_2, b_3, b_4; \theta_1, \theta_2, \theta_3] \quad (3.6)$$

$$= \int_{x_2=-\infty}^{b_2} \int_{x_3=-\infty}^{\frac{b_3-\theta_2 x_2}{\sqrt{1-\theta_2^2}}} \frac{\exp\left(-\frac{\left(x_2^2+x_3^2\right)}{2}\right)}{2\pi} N\left[\frac{b_1 - \theta_1 x_2}{\sqrt{1-\theta_1^2}}\right] N\left[\frac{b_4 - \theta_3\sqrt{1-\theta_2^2}\,x_3 - \theta_3\theta_2 x_2}{\sqrt{1-\theta_3^2}}\right] dx_2 dx_3$$

- When there are five fixing dates, the actual numerical dimension of the function Φ_5 can be brought down from 5 to 3 by using :

$$\Phi_5\left[b_1,b_2,b_3,b_4,b_5;\theta_1,\theta_2,\theta_3,\theta_4\right] \tag{3.7}$$

$$= \int_{x_2=-\infty}^{b_2} \int_{x_3=-\infty}^{\frac{b_3-\theta_2 x_2}{\sqrt{1-\theta_2^2}}} \int_{x_4=-\infty}^{\frac{b_4-\theta_3\sqrt{1-\theta_2^2}x_3-\theta_3\theta_2 x_2}{\sqrt{1-\theta_3^2}}} \frac{\exp\left(-\frac{\left(x_2^2+x_3^2+x_4^2\right)}{2}\right)}{(2\pi)^{3/2}} N\left[\frac{b_1-\theta_1 x_2}{\sqrt{1-\theta_1^2}}\right] N\left[\frac{b_5-\theta_4\sqrt{1-\theta_3^2}x_4-\theta_4\theta_3\sqrt{1-\theta_2^2}x_3-\theta_4\theta_3\theta_2 x_2}{\sqrt{1-\theta_4^2}}\right] dx_2 dx_3 dx_4$$

• More generally, when there are over five fixing dates, the actual numerical dimension of the function Φ_n can always be reduced by a factor of 2 by using :

$$\Phi_n\left[b_1,b_2,\ldots,b_{n-1},b_n;\theta_1,\theta_2,\ldots,\theta_{n-2},\theta_{n-1}\right] \tag{3.8}$$

$$= \int_{x_2=-\infty}^{b_2} \int_{x_3=-\infty}^{b_3} \ldots \int_{x_{n-1}=-\infty}^{b_{n-1}} \frac{\exp\left(-\frac{x_2^2}{2}-\frac{1}{2\left(1-\theta_2^2\right)}\left(x_3-\theta_2 x_2\right)^2 \ldots -\frac{1}{2\left(1-\theta_{n-2}^2\right)}\left(x_{n-1}-\theta_{n-2}x_{n-2}\right)^2\right)}{\prod_{i=2}^{n-2}\sqrt{\left(1-\theta_i^2\right)}\,(2\pi)^{\frac{n-2}{2}}} N\left[\frac{b_1-\theta_1 x_2}{\sqrt{1-\theta_1^2}}\right] N\left[\frac{b_n-\theta_{n-1}x_{n-1}}{\sqrt{1-\theta_{n-1}^2}}\right] dx_2 dx_3 \ldots dx_{n-1}$$

The identities (3.5) – (3.8) can be obtained using tedious algebra and their proof is therefore omitted. The quality of numerical integration using (3.5) – (3.8) was tested up to six fixing dates. That would be the required dimension for the valuation of a three-year expiry cliquet option with semi-annual observation points or a six-month expiry cliquet option with monthly observation frequency, for instance. The identities (3.5) – (3.8) were implemented using a simple 16-point Gauss-Legendre quadrature rule. In every dimension, 500 option prices were computed by means of Proposition 1, Proposition 2 and Proposition 3, with randomly drawn parameters. The results were compared with those obtained by a Monte Carlo simulation using antithetic variates and the Mersenne Twister random number generator. The main findings are summarized in Tables 1, 2, 3 and 4. In terms of efficiency, it always takes less than one second to compute the option prices on an ordinary personal computer, which is very quick in absolute terms and dramatically more efficient than simulation techniques. It must be pointed out that the efficiency gains are even greater when it comes to the computation of the option sensitivities or greeks. Moreover, there is a clear pattern of linear convergence of the Monte Carlo estimates to the analytical values as more and more simulations are performed in all the tested dimensions. This high quality of numerical integration is not surprising, given the smoothness of the integrands in every dimension. Cases of numerical instability might arise only for extremely high values of the correlation coefficients inside the integrands. After a little numerical experiment, it was found that problems were encountered for values above 99.5%. But such values would come up only for very weird contract specifications, such a single fixing date one day before expiry on a two-year option, and these would actually never be met in the markets. Besides, alternative, more sophisticated quadrature rules that handle almost singular points could then be resorted to.

Thus, in low dimensions, the analytical formulae provided in this paper give very efficient and accurate numerical results, while the computational time required for a Monte Carlo simulation to produce reasonably accurate approximations is clearly not satisfactory for practical purposes. As more and more fixing dates are added, the quality of a plain Gauss-Legendre implementation of (3.8) will deteriorate. One solution is then to implement an adaptive Gauss-Legendre quadrature based, for instance, on a Kronrod rule. There is plenty of scientific software available for that matter so that it is not necessary to know the technical details. Another, more powerful, solution is to notice that the function Φ that needs to be computed in increasing dimension has the attractive feature that it matches the special structure of Gaussian convolutions handled by the extremely efficient Broadie-Yamamoto algorithm (Broadie and Yamamoto, 2005). Using the latter will ensure that the above stated Proposition 3 can be implemented with confidence in high dimensions. For evidence about the power of the mentioned algorithm, the reader is referred to the above cited original paper.

Table 1

Cliquet option values with three non-uniformly spaced fixing dates [a]

	Average computational time	Average difference with analytical formula	Maximum difference with analytical formula
Analytical formula	0.15	0	0
Monte Carlo 100,000 simulations	0.82	0.63	0.96
Monte Carlo 1,000,000 simulations	5.58	0.19	0.56
Monte Carlo 10,000,000 simulations	52.6	0.025	0.161

[a] The analytical formula used is Proposition 1. The function Φ_3 in Proposition 1 is computed using identity (3.5) along with a 16-point Gauss-Legendre quadrature. The reported averages were computed out of a sample of 500 theoretical option values with randomly drawn parameters. The average computational time is measured in seconds. The average difference between the Monte Carlo approximations and the analytical prices is reported as a percentage of the option prices. So is the maximum difference too.

Table 2

Cliquet option with four non-uniformly spaced fixing dates [b]

	Average computational time	Average difference with analytical formula	Maximum difference with analytical formula
Analytical formula	0.26	0	0
Monte Carlo 100,000 simulations	1.07	0.45	1.05
Monte Carlo 1,000,000 simulations	7.88	0.17	0.46
Monte Carlo 10,000,000 simulations	74.81	0.029	0.213

[b] The analytical formula used is Proposition 2. The function Φ_4 in Proposition 2 is computed using identity (3.6) along with a 16-point Gauss-Legendre quadrature.

Table 3

Cliquet option with five non-uniformly spaced fixing dates [c]

	Average computational time	Average difference with analytical formula	Maximum difference with analytical formula
Analytical formula	0.38	0	0
Monte Carlo 100,000 simulations	1.24	0.51	0.82
Monte Carlo 1,000,000 simulations	9.85	0.22	0.63
Monte Carlo 10,000,000 simulations	95.32	0.042	0.176

[c] The analytical formula used is Proposition 3. The function Φ_5 in Proposition 3 is computed using identity (3.7) along with a 16-point Gauss-Legendre quadrature.

Table 4

Cliquet option with six non-uniformly spaced fixing dates [d]

	Average computational time	Average difference with analytical formula	Maximum difference with analytical formula
Analytical formula	0.89	0	0
Monte Carlo 100,000 simulations	1.45	0.54	0.91
Monte Carlo 1,000,000 simulations	12.16	0.26	0.49
Monte Carlo 10,000,000 simulations	123.66	0.038	0.184

[d] The analytical formula used is Proposition 3. The function Φ_6 in Proposition 3 is computed using identity (3.8) along with a 16-point Gauss-Legendre quadrature.

IV. APPENDIX : PROOF OF PROPOSITION 1

The dynamics of the underlying asset S under the risk-neutral measure Q are driven by the classical geometric Brownian motion:

$$dS(t) = (r - d)S(t)dt + \sigma S(t)dB(t) \tag{4.1}$$

Where r is the constant riskless rate, σ is the constant volatility of S, d is a constant payout rate on S and B(t) is a standard Brownian motion.

Following the risk-neutral valuation method (Harrison and Kreps, 1979; Harrison and Pliska, 1981), the no-arbitrage value, at time t_0, of a cliquet call option with three fixing dates t_1, t_2, t_3 and expiry t_3, is given by:

$$\begin{aligned} &C(S(t_0), K, r, s, d, t_1, t_2, t_3) \\ &= \exp(-r \times t_3) \times E_Q[(S(t_1) \vee S(t_2) \vee S(t_3) - K)^+ | S(t_0)] \end{aligned} \tag{4.2}$$

where E_Q is the expectation operator under the risk-neutral probability measure Q. Breaking down the various possible outcomes, the right-hand side of (4.2) can be expanded as follows :

$$\exp(-r \times t_3) \times E_Q[(S(t_1)-K) \times I_1 + (S(t_2)-K) \times I_2 + (S(t_3)-K) \times I_3 | S(t_0)] \quad (4.3)$$

with

$$I_1 = I\{S(t_1) > K, S(t_1) > S(t_2), S(t_1) > S(t_3)\} \quad (4.4)$$

$$I_2 = I\{S(t_2) > K, S(t_2) > S(t_1), S(t_2) > S(t_3)\} \quad (4.5)$$

$$I_3 = I\{S(t_3) > K, S(t_3) > S(t_1), S(t_3) > S(t_2)\} \quad (4.6)$$

where I is the indicator function.

Using Girsanov's theorem, (4.3) becomes:

$$\begin{aligned} &\exp(-\delta t_1 - r(t_3 - t_1)) \times S(t_0) \times E_{Q(1)}[I_1 | S(t_0)] \\ &+ \exp(-\delta t_2 - r(t_3 - t_2)) \times S(t_0) \times E_{Q(2)}[I_2 | S(t_0)] \\ &+ \exp(-\delta t_3) \times S(t_0) \times E_{Q(3)}[I_3 | S(t_0)] - \exp(rt_3) \times K \times E_Q[I_1 + I_2 + I_3 | S(t_0)] \end{aligned} \quad (4.7)$$

where $Q^{(1)}$, $Q^{(2)}$ and $Q^{(3)}$ are three measures equivalent to Q whose Radon-Nikodym derivatives are given by :

$$\frac{dQ^{(i)}}{dQ} | F_{t_i} = \exp\left(-\frac{\sigma^2}{2} t_i + \sigma B(t_i)\right), i \in \{1,2,3\} \quad (4.8)$$

where F_t is the natural filtration of B(t).

Thus, it suffices to compute the expectations of I_1, I_2 and I_3 under Q. A classical change of drift from $\mu = (r - d - \sigma^2/2)t_i$ to $\mu = (r - d + \sigma^2/2)t_i$ will provide the same expectations under $Q^{(i)}$.

Since the logarithm is a monotonically increasing function, the functions I_1, I_2 and I_3 are equal to:

$$I_1 = I\{X(t_1) > k, X(t_1) > X(t_2), X(t_1) > X(t_3)\} \quad (4.9)$$

$$I_2 = I\{X(t_2) > k, X(t_2) > X(t_1), X(t_2) > X(t_3)\} \quad (4.10)$$

$$I_3 = I\{X(t_3) > k, X(t_3) > X(t_1), X(t_3) > X(t_2)\} \quad (4.11)$$

where $X(t) = \ln(S(t)/S(t_0))$ and $k = \ln(K/S(t_0))$

Next, by definition of Brownian motion, the random variables $X(t_1)$, $X(t_2) - X(t_1), X(t_3) - X(t_1)$ and $X(t_3) - X(t_2)$ can be written as follows:

$$X(t_1) = \mu t_1 + \sigma\phi_1\sqrt{t_1} \tag{4.12}$$

$$X(t_2) - X(t_1) = \mu(t_2 - t_1) + \sigma\phi_2\sqrt{t_2 - t_1} \tag{4.13}$$

$$X(t_3) - X(t_1) = \mu(t_3 - t_1) + \sigma\left(\phi_2\sqrt{t_2 - t_1} + \phi_3\sqrt{t_3 - t_2}\right) \tag{4.14}$$

$$X(t_3) - X(t_2) = \mu(t_3 - t_2) + \sigma\phi_3\sqrt{t_3 - t_2} \tag{4.15}$$

where ϕ_1, ϕ_2 and ϕ_3 are three mutually independent standard normal random variables with unit variance.

To calculate $E_Q[I_1|S(t_0)]$, one can notice, by the independence of Brownian increments, that the variable $X(t_1)$ is independent from the variables $X(t_2) - X(t_1)$ and $X(t_3) - X(t_1)$. Then, using (4.12) – (4.14), a little calculation produces:

$$E_Q[I_1|S(t_0)]$$
$$= \Phi_1\left[\frac{\ln(S/K) + \mu t_1}{\sigma\sqrt{t_1}}\right]\Phi_2\left[-\frac{\mu(t_2 - t_1)}{\sigma\sqrt{(t_2 - t_1)}}, -\frac{\mu(t_3 - t_1)}{\sigma\sqrt{(t_3 - t_1)}}; \sqrt{\frac{t_2 - t_1}{t_3 - t_1}}\right] \tag{4.16}$$

To compute $E_Q[I_2|S(t_0)]$, one can start by expanding $Q(X(t_2) > k, X(t_2) > X(t_1))$ as follows:

$$Q(X(t_2) > k, X(t_2) > X(t_1))$$
$$= Q(X(t_2) > k, X(t_1) < k) + Q(X(t_2) - X(t_1) > 0)Q(X(t_1) > k) \tag{4.17}$$

Then, using the independence of the variables $X(t_3) - X(t_2)$ and $X(t_2) - X(t_1)$, one can obtain :

$$E_Q[I_2|S(t_0)]$$
$$= \left(\Phi_2\left[\frac{k - \mu t_1}{\sigma\sqrt{t_1}}, \frac{-k + \mu t_2}{\sigma\sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}}\right] + \Phi_1\left[\frac{-k + \mu t_1}{\sigma\sqrt{t_1}}\right]\Phi_1\left[\frac{\mu(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}}\right]\right) \tag{4.18}$$
$$\times \Phi_1\left[\frac{-\mu(t_3 - t_2)}{\sigma\sqrt{(t_3 - t_2)}}\right]$$

Next, the expectation $E_Q[I_3|S(t_0)]$ can be expanded into integral form as follows:

$$E_Q[I_3|S(t_0)] \tag{4.19}$$
$$= \int_{x_1=-\infty}^{x_3}\int_{x_2=-\infty}^{x_3}\int_{x_3=-\infty}^{k} Q(X(t_1) \in dx_1, X(t_2) \in dx_2, X(t_3) \in dx_3)\,dx_3\,dx_2\,dx_1$$

By the Markov property of Brownian motion, we have:

$$
\begin{aligned}
&Q\left(X(t_1)\in dx_1, X(t_2)\in dx_2, X(t_3)\in dx_3\right) \qquad (4.20)\\
&= Q\left(X(t_1)\right)\times Q\left(X(t_2)\in dx_2 \mid X(t_1)\right)\times Q\left(X(t_3)\in dx_3 \mid X(t_2)\right)\\
&= \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{(1-t_1/t_2)(1-t_2/t_3)}}\times\\
&\exp\left(-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma\sqrt{t_1}}\right)^2-\frac{\left(\frac{x_2-\mu_2}{\sigma\sqrt{t_2}}-\sqrt{\frac{t_1}{t_2}}\frac{x_1-\mu_1}{\sigma\sqrt{t_1}}\right)^2}{2(1-t_1/t_2)}-\frac{\left(\frac{x_3-\mu_3}{\sigma\sqrt{t_3}}-\sqrt{\frac{t_2}{t_3}}\frac{x_2-\mu_2}{\sigma\sqrt{t_2}}\right)^2}{2(1-t_2/t_3)}\right)
\end{aligned}
$$

Then, performing the necessary calculations, one can obtain the following result:

$$
\begin{aligned}
&E_Q[I_3 \mid S(t_0)] \qquad (4.21)\\
&= \Phi_3\left[\frac{-k+\mu_3}{\sigma\sqrt{t_3}},\frac{\mu(t_3-t_1)}{\sigma\sqrt{(t_3-t_1)}},\frac{\mu(t_3-t_2)}{\sigma\sqrt{(t_3-t_2)}};\sqrt{1-\frac{t_1}{t_3}},\sqrt{\frac{t_3-t_2}{t_3-t_1}}\right]
\end{aligned}
$$

This completes the proof for a cliquet call option with three fixing dates t_1, t_2, t_3 and expiry t_3. The same method can be applied to value cliquet options with a greater number of fixing dates, by using the following generalization of (4.21) for $n \geq 4$:

$$
\begin{aligned}
&Q\left(X(t_n)>k, X(t_n)>X(t_1), X(t_n)>X(t_2),\ldots,X(t_n)>X(t_{n-1})\right)\\
&= \Phi_n\left[\begin{array}{l}\frac{-k+\mu_n}{\sigma\sqrt{t_n}},\frac{\mu(t_n-t_1)}{\sigma\sqrt{(t_n-t_1)}},\frac{\mu(t_n-t_2)}{\sigma\sqrt{(t_n-t_2)}},\ldots,\frac{\mu(t_n-t_{n-1})}{\sigma\sqrt{(t_n-t_{n-1})}};\\ \sqrt{1-\frac{t_1}{t_n}},\sqrt{\frac{t_n-t_2}{t_n-t_1}},\sqrt{\frac{t_n-t_3}{t_n-t_2}},\ldots,\sqrt{\frac{t_n-t_{n-1}}{t_n-t_{n-2}}}\end{array}\right] \qquad (4.22)
\end{aligned}
$$

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