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On Hedge Parameters of currency Options

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ABSTRACT

Currency options differ from stock options because there are two (domestic and foreign) interest rates involved. In this paper, we explicitly derive the Black-Scholes option pricing formula for currency options. The partial derivatives of the option price, also known as hedge parameters, give us insight into the effect of option variables on the option's cost. We derive and mathematically analyze the hedge parameters to interpret their role and meaning in financial securities. This paper is expository in style about currency options for audiences of a wide range, including students majoring in mathematical finance at business schools.

JEL Classifications: G130

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I. INTRODUCTION

Since the introduction of the Black-Scholes option pricing formula in 1973, many variations have evolved along with the emergence of new financial derivatives, and the Black-Scholes option pricing formula had been a symbol of the fancy mathematics used in the financial industry. During the past decade, the financial community paid more attention to risk management and created many elaborate financial derivatives, which caused an increase in the demand for high-level mathematics and a deep understanding of finance. As a result, interdisciplinary research and education in mathematics and financial economics have evolved in the nineties. Reflecting the popularity and the marketability of the subject these days, many people, both academics and practitioners, want to study financial mathematics, yet the resources have not kept the demand.

This paper is, as already noted, expository writing about currency options for audiences of a wide range, including students majoring in mathematical finance at business schools. While the main text contains well-established results for international finance, the Appendix contains all the mathematical nuts and bolts that do not exist in any literature, at least to the authors' knowledge.1

In the main text, we explain the financial meaning of European options on foreign currency and the derivation of the Black-Scholes option pricing formula. Also presented is how the foreign exchange market makes the put-call parity of currency options different from stock options. Finally, the hedge parameters of options, also known as the Greek letters, are defined, and their mathematical formulas are shown.

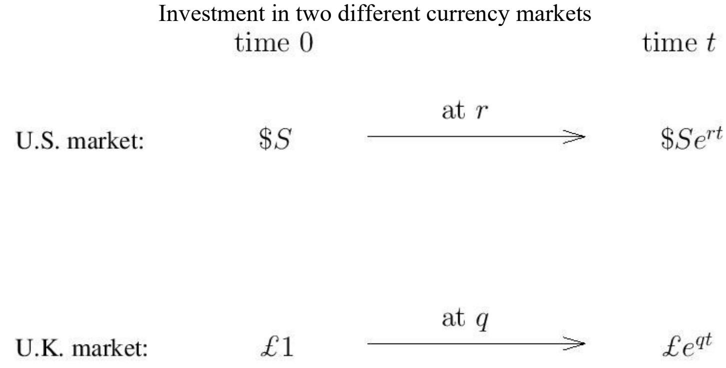
In the Appendix, complete mathematical derivation and analysis of all the main text results are offered, and it is this part that people in the financial community would find useful. On the other hand, mathematicians would benefit from the financial concept in the main text. Currency options have the same mathematical structure as options on continuous-dividend-yielding stock and futures options as seen in Hull (2017), therefore, work in this paper applies to these two options with only a slight modification.

II. CONCEPT AND RESULT IN FINANCE

A. What is an Option?

A currency option is a financial derivative which gives its holder a right to purchase a certain number of units of a foreign currency at a pre-determined price. We use the term spot price (or spot exchange rate) expressing the price of a foreign currency in terms of the U. S. dollar, our domestic currency throughout this paper. To value the cost of a currency option, we first need to know how the present exchange rate (the spot price) evolves as time goes by. Consider the following example. Assume that an investor invests £1 each in two markets, the U.K. and in the U.S. where the risk-free interest rates are q and r, respectively. Without loss of generality, no arbitrage opportunity is assumed.[2] In the U.S. market at the time of the investment, the investor purchases S units of U.S. dollars with the £1, which means the time 0 spot price of the British pound is \$$S$. The investor then lets the two investments grow under the respective risk-free interest rates q and r. After time t, the \$$S$ has grown to \$$Se^{rt}$ in the U.S. market and the £1 to £e^{qt} in the U.K. market. Since the two initial investments of \$$S$ and £1 are equivalent in value at time 0 and no arbitrage is assumed, their values in time t should remain the same, therefore $\$Se^{rt} = £e^{qt}$. See Figure 1.

Figure 1
Investment in two different currency markets



This implies that the spot exchange rate at time

$$S^*(t) = \frac{Se^{rt}}{e^{qt}} = Se^{(r-q)t} \tag{1}$$

This is the well-known interest rate parity formula. The mathematical model we adopt here for $S^*(t)$ is a continuous-time lognormal approach.[3] It assumes the change of the spot price during Δt is normally distributed with mean $(r-q)\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$.

Mathematically it is represented by the exchange rate differential equation

$$dS^* = (r-q)S^*dt + \sigma S^*dB \tag{2}$$

where B is a normal random variable with mean 0 and variance t. The risk-neutral solution of this exchange rate is

$$S^*(t) \quad = Se^{\left\{\left(r-q-\frac{\sigma^2}{2}\right)t+\sigma B\right\}} \quad = Se^{-qt}\, e^{\left\{\left(r-\frac{\sigma^2}{2}\right)t+\sigma B\right\}} \tag{3}$$

where S is the spot price at the present time 0 and this can be rewritten as

$$S^*(t) = Se^{-qt}\, e^{\left\{\left(r-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}Z\right\}} \tag{4}$$

Here Z is a standard normal random variable (mean 0 and variance 1). From Equation (2), we will first derive the Black-Scholes option pricing formula for European call options on currency and the partial derivatives of the option cost. Since we are dealing with another interest rate q in addition to r, there will be two partial derivatives with respect to interest rate as opposed to the case of stock options.

B. Black-Scholes Option Pricing Formula

In this section, we derive the Black-Scholes option pricing formula for currency options. We must know where it starts and how it is derived, for parts of the derivation process are very useful in getting the partial derivatives, which directly leads to this paper's result. Recall that a random variable Z is called standard normal if its distribution function N is

$$P(Z \le t) = N(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{x^2}{2}} dx \tag{5}$$

This has the property $N(x) = 1 - N(-x)$ for $0 \le x < \infty$. Recall that the value of a European stock option is the present value of the expected value of the maximum of $S(t) - K$ where $S(x)$ is the stock price at the expiration time t and K is the strike price. In evaluating the option values, the stock price model follows the continuous time lognormal approach. It assumes the change of the stock price during Δt is normally distributed with mean $r\Delta t$ and standard deviation $\sigma\sqrt{\Delta t}$. This yields the following stock differential equation

$$dS = rSdt + \sigma SdB \tag{6}$$

where B is a normal random variable with mean 0 and variance t. This the same as our spot exchange rate differential equation (1) except for the expected return part (r vs. $r - q$). Hence we can write the spot exchange rate $S^*(t)$ as

$$S^*(t) = e^{-qt} S(t) \tag{7}$$

where

$$S(t) = S\,\mathrm{e}^{\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma B\right\}} \quad = S\,\mathrm{e}^{\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z\right\}} \tag{8}$$

Using this notation, the present value of a European call option on currency is

$$\begin{aligned} C(S, t, K, \sigma, r, q) &= E\left[e^{-rt}\left(e^{-qt}S(t) - K\right)^+\right] \\ &= E\left[e^{-(r+q)t}\left(S(t) - e^{qt}K\right)^+\right] \\ &= E\left[e^{-(r+q)t}\,\mathrm{m}\ \ \left(S(t) - e^{qt}K, 0\right)\right] \end{aligned} \tag{9}$$

and with some calculation we have the following result.

Theorem 1 *(Black-Scholes Option Pricing Formula for Currency Options)*

The cost of a European call option on the exchange rate which follows a geometric Brownian motion with the domestic risk-free interest rate, foreign risk-free interest rate, and exchange rate's volatility is given as

$$C(S, t, K, \sigma, r, q) = e^{-qt} S N(d_1) - e^{-rt} K N(d_2)$$

where $d_1 = \frac{\ln\frac{S}{K} + \left(r - q + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$, $d_2 = \frac{\ln\frac{S}{K} + \left(r - q - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}$ *and* $N(\cdot)$ *is the standard normal distribution.*

From this call option pricing formula, we can derive its hedge parameters, also known as the Greek letters. They are none other than the partial derivatives of the option price with respect to each variable, which shows how changes in the variables affect the option price. The parameters are defined as

$$\Delta = \frac{\partial C}{\partial S},\ \Gamma = \frac{\partial C}{\partial \Delta},\ \Theta = -\frac{\partial C}{\partial t},\ \nu = \frac{\partial C}{\partial \sigma},\ \rho = \frac{\partial C}{\partial r},\ \rho_f = \frac{\partial C}{\partial q}.$$

The same partial derivatives for put options are defined, so from now on, we specify the option kind for the hedge parameters, such as call-delta (Δ_c), put-rho (ρ_p) etc.

C. Partial Derivatives: Call Options

In this section, we introduce the partial derivatives of the call option value. For the strike price K, we derive both $\frac{\partial C}{\partial K}$ and $\frac{\partial^2 C}{\partial K^2}$, which do not have any Greek titles but hold importance of their own. The result is summarized in Table 1.

Table 1
Call option parameters and formulas

Symbol	Identity	Formula
Δ_c	$\frac{\partial C}{\partial S}$	$e^{-qt}N(d_1)$
Γ_c	$\frac{\partial \Delta_c}{\partial S}$	$\frac{e^{-qt}N'(d_1)}{S\sigma\sqrt{t}}$
θ_c	$-\frac{\partial C}{\partial t}$	$-\frac{SN'(d_1)\sigma e^{-qt}}{2\sqrt{t}} + qSN(d_1)e^{-qt} - re^{-rt}KN(d_2)$
ν_c	$\frac{\partial C}{\partial \sigma}$	$S\sqrt{t}\,N'(d_1)e^{-qt}$
ρ_c	$\frac{\partial C}{\partial r}$	$te^{-rt}KN(d_2)$
$\rho_{f,c}$	$\frac{\partial C}{\partial q}$	$-te^{-qt}SN(d_1)$
	$\frac{\partial C}{\partial K}$	$-e^{-rt}N(d_2)$
	$\frac{\partial^2 C}{\partial K^2}$	$\frac{e^{-rt}N'(d_2)}{\sigma\sqrt{t}K}$

Note: A word of caution is needed here for people with weaker mathematical background, especially students in business programs. Many textbooks of finance and international finance explicitly show the definition of Δ and its value. It is tempting to simply drop S to get Δ from C to get $\frac{\partial}{\partial S}\{e^{-qt}\mathrm{SN}\,(d_1) - e^{-rt}KN(d_2)\} = e^{-qt} \cdot 1 \cdot N(d_1) + 0 = e^{-q}\,N(d_1)$ as is the case in $\frac{d}{dx}(2x) = 2$. This straightforward differentiation is not correct because both d_1 and d_2 are functions of S (and other variables t, K, σ, r, q), so we have to use chain rule to get Δ.

D. Partial Derivatives: Put Options

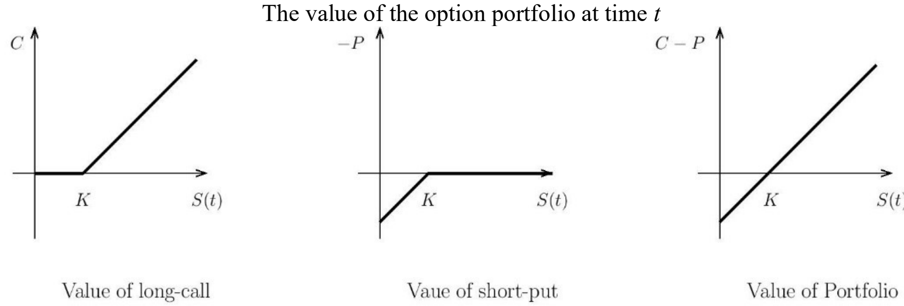
As mentioned earlier, put options have pricing formula and hedge parameters which are defined exactly the same way as those of call options. More precisely,

$$\begin{aligned} P(S,t,K,\sigma,r) &= E\left[e^{-rt}\left(K - e^{-qt}S(t)\right)^{+}\right] \\ &= E[e^{-rt}\,\mathrm{m}\;\;(K - e^{-qt}S(t), 0)] \end{aligned} \tag{10}$$

The rest of the derivation is identical to the call option case shown in Subsection 2.2. Instead of reproducing the calculations all over, we derive the Put-Call Parity of European currency options and use it for put option pricing and its hedge parameters.

First, recall the Put-Call Parity of European stock options. This parity is derived by replicating an investment portfolio that consists of buying one call option and selling one put option (long on call and short on put), which have the same expiration t and the strike price K. At time t, the value of the long-call and short-put are as shown in Figure 2, thus the value of this portfolio at time t is $S(t) - K$ where $S(t)$ is the price of the underlying security at time t.

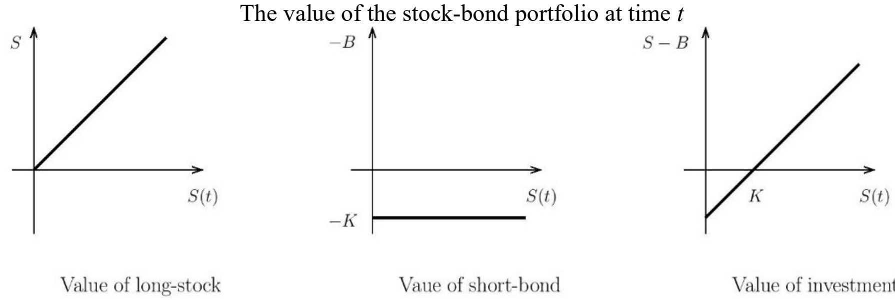
Figure 2
The value of the option portfolio at time *t*



This value is the same as that of another portfolio which consists of a share of the underlying stock and a cash outflow of K. This can be achieved by, at time 0, purchasing a share of the stock at S and selling a bond whose principal is $e^{-rt}K$ and matures in t. At time t the value of this stock-bond portfolio is $S(t) - K$, which is the same as the options portfolio. Assuming no arbitrage opportunity, we conclude that the present (time 0) value of the option portfolio and the stock-bond portfolio are the same. Therefore, we have the famous Put-Call Parity

$$C - P = S - e^{-rt}K \tag{11}$$

Figure 3
The value of the stock-bond portfolio at time t



The Put-Call Parity for currency options can be derived in the same way, except that there are two security markets to be considered, the domestic market and the foreign market. In case of currency options, the first investment is buying and selling the options in the domestic market. The second investment is to buy a e^{-qt} unit of foreign currency in the foreign market and invest it there at the risk-free rate q. In the domestic market at the same time, sell a bond worth $e^{-rt}K$ and maturing at time t and let the proceed grow at the domestic risk-free rate r. When the options expires at time t, either one is in the money and the portfolio value is $S^*(t) - K$ in the domestic currency. The currency-bond portfolio is worth a unit of foreign currency in the foreign market and a cash outflow of K in the domestic market (to pay the bond), which is equivalent to $S^*(t) - K$ in the domestic currency. Therefore, after converting to the present value of the option portfolio in the domestic currency, we have the following Put-Call Parity for currency options,

$$C - P = e^{-qt}S - Ke^{-rt} \tag{12}$$

With the fact $N(x) - 1 = -N(-x)$, the put option becomes

$$\begin{aligned} P(S,t,K,\sigma,r,q) &= C - Se^{-qt} + Ke^{-rt} \\ &= e^{-qt}SN(d_1) - e^{-rt}KN(d_2) - e^{-qt}S + Ke^{-rt} \\ &= Se^{-qt}(N(d_1) - 1) + Ke^{-rt}\big(1 - N(d_2)\big) \\ &= -Se^{-qt}N(-d_1) + Ke^{-rt}N(-d_2) \end{aligned} \tag{13}$$

Using this, for one of the six variables x of P,

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\{C - Se^{-qt} + Ke^{-rt}\} = \frac{\partial C}{\partial x} - \frac{\partial}{\partial x}\{Se^{-qt} + Ke^{-rt}\} \tag{14}$$

and we have the hedge parameters for put options. The result is summarized in Table 2.

Table 2
Put option parameters and formulas

Symbol	Identity	Formula
Δ_p	$\frac{\partial P}{\partial S}$	$e^{-qt}\{N(d_1)-1\}$
Γ_p	$\frac{\partial \Delta_p}{\partial S}$	$\frac{e^{-qt}N(d_1)}{S\sigma\sqrt{t}}$
θ_p	$-\frac{\partial P}{\partial t}$	$-\frac{SN'(d_1)\sigma e^{-qt}}{2\sqrt{t}} - qSN(-d_1)e^{-qt} + re^{-rt}KN(-d_2)$
ν_p	$\frac{\partial P}{\partial \sigma}$	$S\sqrt{t}N'(d_1)e^{-qt}$
ρ_p	$\frac{\partial P}{\partial r}$	$-te^{-rt}KN(-d_2)$
$\rho_{f,p}$	$\frac{\partial P}{\partial q}$	$te^{-qt}SN(-d_1)$
	$\frac{\partial P}{\partial K}$	$-e^{-rt}\{N(d_2)-1\}$
	$\frac{\partial^2 P}{\partial K^2}$	$\frac{e^{-rt}N(d_2)}{\sigma\sqrt{t}K}$

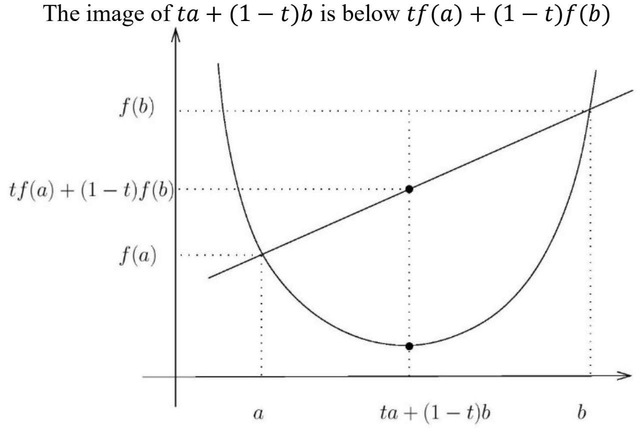
III. PARAMETER CHANGES AND OPTION PRICES

In the previous two sections, we have derived the hedge parameters of European call and put options on currency. In this section, we use them to study how the change of currency option variables affect the price of options. Some variables are straightforwardly related to the option price and they move in the same direction. Some parameters give even more information about the option price. Assume that among the six option variables, the spot price S in the only one that changes and the other five remainfixed, then the call option price is a function of S only. Let C_1 and C_2 be the respective call prices for $S_1 < S_2$. Under certain conditions, it is guaranteed that the option price for S, the average (mean) of S_1 and S_2, isalways less than or equal to the average of C_1 and C_2. This is in fact true for all weighted average S of S_1 and S_2 and the corresponding weighted average C of C_1 and C_2. This property is called convexity. A real valued function $f(x)$ defined on real numberse, $f: \mathbb{R} \to \mathbb{R}$ is said to be convex if for all $a, b \in \mathbb{R}$ and $0 < t < 1$,

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \qquad (15)$$

This means that any point between a and b has its image under the line that joins $f(a)$ and $f(b)$. See Figure 4.

Figure 4
The image of $ta + (1-t)b$ is below $tf(a) + (1-t)f(b)$



This kind of bent-up curve is described as concave up in calculus and every calculus class introduces this concave-up curve in conjunction with the Concavity Test. It says that the graph of a twice differentiable function f is concave up at every point in an interval I if $f''(x) > 0$ for every x in I. With this concavity test, we have the following corollary on the option values. The detail of the proofs can be found in Appendix.

Corollary 1 *The price of a call option* $C(S, t, K, \sigma, r, q)$ *is*

(a) increasing and convex in S
(b) decreasing and convex in K
(c) increasing in r, decreasing in q
(d) increasing in σ
(e) inconclusive in t

and for the put counterpart, the price of a put option $P(S, t, K, \sigma, r, q)$ is

Corollary 2 *The price of a put option* $P(S, t, K, \sigma, r, q)$ *is*

(a) decreasing and convex in S
(b) increasing and convex in K
(c) decreasing in r, increasing in q
(d) increasing in σ
(e) inconclusive in t

Thus, we have the following result:

Table 3
Marginal effect of a parameter change of option prices

Variable		Price Effect on Call
Spot Price S ↑		↑
Exercise Price K ↑	↓	↑
Domestic Interest Rate r ↑	↑	↓
Foreign Interest Rate q ↑	↓	↑
Spot Rate Volatility σ ↑	↑	↑
Time to Maturity t ↑	Ambiguous	Ambiguous

ENDNOTES

1. Ross (2003) has a partial mathematical work for European call options. Chriss (1997) lists only the final formulas without proofs.
2. Even if there were an arbitrage opportunity, it would last a very short time due to the active involvement of arbitrageurs.
3. This is a very popular stock pricing model called Geometric Brownian Motion and it was with this model that the original Black-Scholes option pricing formula was derived. Goodman and Stampfli (2001) extended the technique to currency options.
4. In this paper, our present time is set to be 0 and t is the time left to expiration. Since t decreases as the option gets close to its expiration, we need the negative sign (-) in the definition of Θ. However, to measure the time to expiration, people have also used T-t where T is the option expiration date and t is the present time. This T-t decreases as t increases, so when T-t is used, Θ is defined to be ∂C/∂t without the negative sign.

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APPENDIX: MATHEMATICAL DETAILS

1. Proof of Theorem 1

The value of a currency option is the present value of the expected value of the maximum of $S^*(t) - K = e^{-qt}S(t) - K$,

$$C(S,t,K,\sigma,r,q) = e^{-rt}E[(e^{-q}\ S(t) - K)^+] = e^{-(r+q)t}E[(S(t) - e^{qt}K)^+]$$

where $S(t) = S\exp\left\{\left(r - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z\right\}$. Recall that when X is a random variable with probability density function f(x), the expected value of $g(X)$ is

$$E[g(X)] = \int f(x)g(x)dx.$$

Therefore we have

$$\begin{aligned} C(S,t,K,\sigma,r,q) &= e^{-(r+q)t}\int_{-\infty}^{\infty} \{S\ e^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{t}x} - e^{qt}K\}^+ \cdot \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx \\ &= e^{-(r+q)t}\int_{a}^{\infty} \{S\ e^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{t}x} - e^{qt}K\} \cdot \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx \end{aligned}$$

where a is decided by solving the following inequality for x:

$S\,e^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{t}x} \geq e^{qt}K$ or equivalently $e^{(r-\frac{\sigma^2}{2})t+\sigma\sqrt{t}x} \geq e^{qt}\frac{K}{S}$. This implies $(r-\frac{\sigma^2}{2})t + \sigma\sqrt{t}x \geq \ln\{e^{qt}\frac{K}{S}\} = qt + \ln\frac{K}{S}$, hence $x \geq \frac{qt+\ln\frac{K}{S}-(r-\frac{\sigma^2}{2})t}{\sigma\sqrt{t}} = \frac{\ln\frac{K}{S}+(q-r+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}} = a$.
To handle the maximum value $(\cdot,\cdot)^+$, we opt the indicator random variable I elegantly used in [8],

$$I = \begin{cases} 1 & \text{if } S(t) \geq e^{qt}K \\ 0 & \text{otherwise} \end{cases}$$

Solving $S(t) = S\exp\left\{\left(r-\frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z\right\} > e^{qt}K$ for Z, we get

$$I = \begin{cases} 1 & \text{if } Z \geq \dfrac{\ln\frac{K}{S} + \left(q - r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} \\ 0 & \text{otherwise.} \end{cases}$$

Using this indicator random variable and $(S(t) - e^{qt}K)^+ = I(S(t) - e^{qt}K)$, the call price now becomes

$$\begin{aligned} C(S,t,K,\sigma,r,q) &= e^{-(r+q)t}E[(S(t)-e^{qt}K)^+] = e^{-(r+q)t}E[I(S(t)-e^{qt}K)] \\ &= e^{-(r+q)t}E[IS(t) - Ie^{qt}K] = e^{-(r+q)t}E[IS(t)] - e^{-rt}E[IK] \\ &= e^{-(r+q)t}E[IS(t)] - e^{-rt}KE[I]. \end{aligned}$$

These two components will be used repeatedly when calculating the partial derivatives, so we calculate their respective values and another useful one in the following lemma.

Lemma 1

Let $d_1 = \frac{\ln\frac{S}{K}+\left(r-q+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$ and $d_2 = d_1 - \sigma\sqrt{t}$. Then
(a) $E[IS(t)] = Se^{rt}N(d_1)$
(b) $E[I] = N(d_2)$
(c) $E[IS(t)Z] = Se^{rt}N'(d_1) + \sigma\sqrt{t}N(d_1)$

Proof. Recall that $I = 1$ if $Z \geq \frac{\ln\frac{K}{S}+\left(q-r+\frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = a$. Also

$$d_2 = d_1 - \sigma\sqrt{t} = \frac{\ln\frac{S}{K} + \left(r - q - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}} = -a$$

For part (a),

$$\begin{aligned}
E[IS(t)] &= \int_a^\infty Se^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}x}\cdot\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\\
&= Se^{\left(r-\frac{\sigma^2}{2}\right)t}\int_a^\infty e^{\sigma\sqrt{t}x}\cdot\frac{e^{\frac{x^2}{2}}}{\sqrt{2\pi}}dx\\
&= Se^{\left(r-\frac{\sigma^2}{2}\right)t}\int_a^\infty e^{-\frac{x^2}{2}+\sigma\sqrt{t}x}\frac{1}{\sqrt{2\pi}}dx\\
&= Se^{\left(r-\frac{\sigma^2}{2}\right)t}\int_a^\infty e^{-\frac{1}{2}(x-\sigma\sqrt{t})^2+\frac{\sigma^2 t}{2}}\frac{1}{\sqrt{2\pi}}dx\\
&= Se^{\left(r-\frac{\sigma^2}{2}\right)t}\cdot e^{\frac{\sigma^2 t}{2}}\int_a^\infty e^{-\frac{1}{2}(x-\sigma\sqrt{t})^2}\frac{1}{\sqrt{2\pi}}dx\\
&= Se^{rt}\int_b^\infty e^{-\frac{1}{2}y^2}\frac{1}{\sqrt{2\pi}}dy\ (y = x-\sigma\sqrt{t}, b = a-\sigma\sqrt{t})\\
&= Se^{rt}(1-N(b))\\
&= Se^{rt}\big(1-N(-d_1)\big) = Se^{rt}N(d_1)\ \ \text{(because } b = -d_1\text{)}.
\end{aligned}$$

For part (b),

$$E[I] = \int_a^\infty \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = \int_{-d_2}^\infty \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = 1-N(-d_2) = N(d_2).$$

For part (c),

$$\begin{aligned}
E[IS(t)Z] &= \int_a^\infty Se^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}x}x\cdot\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx\\
&= Se^{\left(r-\frac{\sigma^2}{2}\right)t}\int_a^\infty \frac{1}{\sqrt{2\pi}}e^{\frac{x^2}{2}+\sigma\sqrt{t}x}xdx\\
&= Se^{\left(r-\frac{\sigma^2}{2}\right)t}\int_a^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\sigma\sqrt{t})^2+\frac{\sigma^2 t}{2}}xdx\\
&= Se^{rt}\int_a^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sigma\sqrt{t})^2}{2}}xdx.
\end{aligned}$$

With the same y and b as in part (a) and using the fact

$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$,

$$\begin{aligned}
E[IS(t)Z] &= Se^{rt}\int_{a-\sigma\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}(y+\sigma\sqrt{t})dy\\
&= Se^{rt}\left[-\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\Big|_b^\infty + \sigma\sqrt{t}N(-b)\right]\\
&= Se^{rt}\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} + \sigma\sqrt{t}N(d_1)\right]\\
&= Se^{rt}\big[N'(d_1)+\sigma\sqrt{t}N(d_1)\big].
\end{aligned}$$

Hence

$$C(S,t,K,\sigma,r,q) = e^{-(r+q)t}Se^{rt}N(d_1) - e^{-rt}KN(d_2)$$
$$= e^{-qt}SN(d_1) - e^{-rt}KN(d_2).$$

2. Hedge Parameters for Call Option: Table 1

When x is one of the six variables of

$$C(S,t,K,\sigma,r,q) = E\left[e^{-(r+q)t}I(S(t) - e^{qt}K)\right]$$

where $S(t) = Se^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}z}$,

$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}\left[Ee^{-(r+q)t}I(S(t) - e^{qt}K)\right]$$
$$= E\left[\frac{\partial}{\partial x}\left\{e^{-(r+q)t}I(S(t) - e^{qt}K)\right\}\right]$$
$$= E\left[I\frac{\partial}{\partial x}\left\{e^{-(r+q)t}(S(t) - e^{qt}K)\right\}\right].$$

The output of I is either 1 or 0 when $S(t) \neq e^{qt}K$ and $P\{S(t) = e^{qt}K\} = 0$, hence I does not contribute anything to the expectation. By replacing the x by the six independent variables of C. we can derive the hedge parameters.

(a) Δ_c: $\frac{\partial}{\partial S}\left\{e^{-(r+q)t}(S(t) - e^{qt}K)\right\} = e^{-(r+q)t}\frac{\partial}{\partial S}S(t) - 0 = e^{-(r+q)t}\frac{S(t)}{S}.$

Also $\frac{\partial}{\partial S}S(t) = e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}z} = \frac{S(t)}{S}$, therefore,

$$E\left[I\frac{\partial}{\partial S}\left\{e^{-(r+q)t}(S(t) - e^{qt}K)\right\}\right]$$
$$= E\left[Ie^{-(r+q)t}\frac{S(t)}{S}\right] = \frac{e^{-(r+q)t}}{S}E[IS(t)]$$
$$= \frac{e^{-(r+q)t}}{S}Se^{rt}N(d_1) = e^{-q}\ N(d_1).$$

(b) Γ_c: by chain rule,

$$\frac{\partial}{\partial S}N(d_1) = N'(d_1)\frac{\partial}{\partial S}\left\{\frac{\log\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right\} = N'(d_1)\frac{1}{S\sigma\sqrt{t}},$$

therefore $\Gamma_c = \frac{e^{-qt}N(d_1)}{S\sigma\sqrt{t}}$.

(c) θ_c: $\frac{\partial C}{\partial t} = -E\left[I\frac{\partial}{\partial t}\left\{e^{-(r+q)t}(S(t) - e^{qt}K)\right\}\right]$, and

$$\frac{\partial}{\partial t}\{e^{-(r+q)t}(S(t)-e^{qt}K)\}=\frac{\partial}{\partial t}\{e^{-(r+q)t}S(t)-e^{-r}\ K\}$$
$$=-(r+q)e^{-(r+q)t}S(t)+e^{-(r+q)t}\frac{\partial}{\partial t}\left\{Se^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma z\sqrt{t}}\right\}-\frac{\partial}{\partial t}\{e^{-rt}K\}$$
$$=-(r+q)e^{-(r+q)t}S(t)+e^{-(r+q)t}Se^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma z\sqrt{t}}\left(r-\frac{\sigma^2}{2}+\frac{\sigma}{2\sqrt{t}}Z\right)+Kre^{-rt}$$
$$=-(r+q)e^{-(r+q)t}S(t)+e^{-(r+q)t}S(t)\left(r-\frac{\sigma^2}{2}+\frac{\sigma}{2\sqrt{t}}Z\right)+Kre^{-r}$$
$$=e^{-(r+q)t}S(t)\left(-q-\frac{\sigma^2}{2}+\frac{\sigma}{2\sqrt{t}}Z\right)+Kre^{-rt},$$

therefore

$$-\frac{\partial C}{\partial t}\quad=-E\left[I\left\{e^{-(r+q)t}S(t)\left(-q-\frac{\sigma^2}{2}+\frac{\sigma}{2\sqrt{t}}Z\right)+Kre^{-rt}\right\}\right]$$
$$=e^{-(r+q)t}\left(q+\frac{\sigma^2}{2}\right)E[IS(t)]-e^{-(r+q)t}\frac{\sigma}{2\sqrt{t}}E[IS(t)Z]-re^{-rt}E[IK]$$
$$=e^{-(r+q)t}\left(q+\frac{\sigma^2}{2}\right)Se^{rt}N(d_1)-e^{-(r+q)t}\frac{\sigma}{2\sqrt{t}}Se^{rt}\left\{N'(d_1)+\sigma\sqrt{t}N(d_1)\right\}-re^{-r}\ KN(d_2)$$
$$=e^{-q}\ \left(q+\frac{\sigma^2}{2}\right)SN(d_1)-e^{-qt}\frac{\sigma}{2\sqrt{t}}SN'(d_1)-e^{-qt}S\frac{\sigma}{2\sqrt{t}}\sigma\sqrt{t}N(d_1)-re^{-rt}KN(d_2)$$
$$=e^{-qt}\left(q+\frac{\sigma^2}{2}-\frac{\sigma^2}{2}\right)SN(d_1)-e^{-qt}\frac{\sigma}{2\sqrt{t}}SN'(d_1)-re^{-rt}KN(d_2)$$
$$=-\frac{SN'(d_1)\sigma e^{-qt}}{2\sqrt{t}}+qSN(d_1)e^{-q}\ -re^{-r}\ KN(d_2).$$

(d) ν_c: $\frac{\partial c}{\partial\sigma}=E\left[I\frac{\partial}{\partial\sigma}\{e^{-(r+q)t}(S(t)-e^{qt}K)\right\}$, and

$$\frac{\partial}{\partial\sigma}\{e^{-(r+q)t}(S(t)-e^{qt}K)\}\quad=e^{-(r+q)t}\frac{\partial}{\partial\sigma}\left\{Se^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}z}\right\}-0$$
$$=e^{-(r+q)t}Se^{\left(r\frac{\sigma^2}{2}\right)t+\sigma\sqrt{t}z}\cdot\left(\frac{-2\sigma}{2}t+\sqrt{t}Z\right)$$
$$=e^{-(r+q)t}S(t)(-\sigma t+\sqrt{t}Z).$$

Therefore,

$$-\frac{\partial C}{\partial\sigma}=E\left[Ie^{-(r+q)t}S(t)(-\sigma t+\sqrt{t}Z)\right]$$
$$=-e^{-(r+q)t}\sigma tE[IS(t)]+e^{-(r+q)t}\sqrt{t}E[IS(t)Z]$$
$$=-e^{-(r+q)t}\sigma tSe^{rt}N(d_1)+e^{-(r+q)t}\sqrt{t}Se^{rt}\left\{N'(d_1)+\sigma\sqrt{t}N(d_1)\right\}$$
$$=-e^{-qt}\sigma tSN(d_1)+e^{-qt}\sqrt{t}SN'(d_1)+e^{-qt}S\sigma t\ N(d_1)$$
$$=S\sqrt{t}N'(d_1)e^{-qt}.$$

(e) ρ_c and $\rho_{f,c}$: for ρ_c, $\frac{\partial C}{\partial r}=E\left[I\frac{\partial}{\partial r}\{e^{-(r+q)t}(S(t)-e^{qt}K)\right\}$, and

$$\frac{\partial}{\partial r}\{e^{-(r+q)t}(S(t)-e^{qt}K)\} = -te^{-(r+q)t}(S(t)-e^{qt}K)+e^{-(r+q)t}S(t)t$$
$$= e^{-(r+q)t}\{-tS(t)+te^{qt}K+tS(t)\}$$
$$= te^{-rt}K.$$

Therefore,

$$-\frac{\partial C}{\partial r} = E\left[\,Ite^{-rt}K\right] = te^{-rt}E[IK] = te^{-rt}KN(d_2).$$

For $\rho_{f,c}$, $\frac{\partial C}{\partial q} = E\left[I\frac{\partial}{\partial q}\{e^{-(r+q)t}(S(t)-e^{qt}K)\}\right]$, and

$$\frac{\partial}{\partial q}\{e^{-(r+q)t}(S(t)-e^{qt}K)\} = -te^{-(r+q)t}(S(t)-e^{qt}K)+e^{-(r+q)t}(0-te^{qt}K)$$
$$= e^{-(r+q)t}\{-tS(t)+te^{qt}K-te^{qt}K\}$$
$$= -te^{-(r+q)t}S(t).$$

Therefore

$$-\frac{\partial C}{\partial q} = E[I(-te^{-(r+q)t}S(t))] = -te^{-(r+q)t}E[IS(t)]$$
$$= -te^{-(r+q)t}Se^{rt}N(d_1) = -e^{-q}\,tSN(d_1).$$

(f) $\frac{\partial C}{\partial K}$: For K, $\frac{\partial C}{\partial K} = E[I\frac{\partial}{\partial K}\{e^{-(r+q)t}(S(t)-e^{qt}K)\}]$, and

$$\frac{\partial}{\partial K}\{e^{-(r+q)t}(S(t)-e^{qt}K)\} = e^{-(r+q)t}(-e^{qt}) = -e^{-rt},$$

therefore $\frac{\partial C}{\partial K} = E[I\{-e^{-rt}\}] = -e^{-rt}E[I] = -e^{-r}\,N(d_2)$.

(g) $\frac{\partial^2 C}{\partial K^2}$: for the second partial derivative with respect to K,

$$\frac{\partial}{\partial K}\{-e^{-rt}N(d_2)\} = -e^{-rt}N'(d_2)\frac{\partial}{\partial K}(d_2)$$
$$= -\;e^{-rt}N'(d_2)\frac{\partial}{\partial K}\{\frac{\ln S-\ln K+(r-q-\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\}$$
$$= \frac{e^{-rt}N'(d_2)}{\sigma\sqrt{t}K}.$$

3. Hedge Parameters for Put Option: Table 2

Throughout the proof we use the Put-Call Parity,

$$\frac{\partial P}{\partial x} = \frac{\partial C}{\partial x} - \frac{\partial}{\partial x}\{Se^{-qt}+Ke^{-rt}\}.$$

(a) Δ_p: $\frac{\partial P}{\partial S} = e^{-qt}N(d_1) - \frac{\partial}{\partial S}\{Se^{-qt}-Ke^{-rt}\} = e^{-qt}\{N(d_1)-1\}$.

(b) Γ_p: just the derivative of (a) with respect to S, and use the fact $e^{-q}\;\frac{\partial}{\partial S}N(d_1) = e^{-qt}N'(d_1)\frac{1}{S\sigma\sqrt{t}}$ (this is Γ_c), hence

$$\frac{\partial^2 P}{\partial S^2} = \frac{\partial}{\partial S}e^{-qt}\{N(d_1)-1\} = e^{-qt}\frac{\partial}{\partial S}N(d_1) - 0 = \frac{e^{-qt}N'(d_1)}{S\sigma\sqrt{t}}.$$

(c) θ_p :

$$
\begin{aligned}
-\frac{\partial P}{\partial t} &= -\frac{\partial C}{\partial t} + \frac{\partial}{\partial t}\{Se^{-qt}\} - \frac{\partial}{\partial t}\{Ke^{-r}\ \} \\
&= -\frac{SN'(d_1)\sigma e^{-qt}}{2\sqrt{t}} + qSN(d_1)e^{-qt} - re^{-rt}KN(d_2) - qSe^{-qt} + rKe^{-rt} \\
&= -\frac{SN'(d_1)\sigma e^{-q}}{2\sqrt{t}} + qSe^{-qt}\{N(d_1) - 1\} - re^{-rt}K\{N(d_2) - 1\} \\
&= -\frac{SN'(d_1)\sigma e^{-qt}}{2\sqrt{t}} - qSe^{-qt}N(-d_1) + rKe^{-rt}N(-d_2).
\end{aligned}
$$

(d) $\nu_p : \frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma} - \frac{\partial}{\partial \sigma}\{Se^{-qt} - Ke^{-rt}\} = \frac{\partial C}{\partial \sigma} = S\sqrt{t}N'(d_1)e^{-qt}$.

(e) ρ_p and $\rho_{f,p}$:

$$
\begin{aligned}
\frac{\partial P}{\partial r} &= \frac{\partial C}{\partial r} - \frac{\partial}{\partial r}\{Se^{-qt} - Ke^{-r}\ \} = te^{-rt}KN(d_2) - te^{-rt}K \\
&= te^{-r}\ K\{N(d_2) - 1\} = -te^{-rt}KN(-d_2),
\end{aligned}
$$

therefore

$$
\begin{aligned}
\frac{\partial P}{\partial q} &= \frac{\partial C}{\partial q} - \frac{\partial}{\partial q}\{Se^{-qt} + Ke^{-rt}\} = -te^{-qt}SN(d_1) + tSe^{-qt} \\
&= te^{-q}\ S\{1 - N(d_1)\} = te^{-qt}SN(-d_1).
\end{aligned}
$$

(f) $\frac{\partial P}{\partial K}$:

$$
\frac{\partial P}{\partial K} = \frac{\partial C}{\partial K} - \frac{\partial}{\partial K}\{Se^{-qt} - Ke^{-rt}\} = -e^{-rt}(N(d_2) - 1).
$$

(g) $\frac{\partial^2 P}{\partial K^2}$:

$$
\frac{\partial^2 P}{\partial K^2} = \frac{\partial}{\partial K}(-e^{-rt}N(d_2) + e^{-r}\ ) = \frac{e^{-rt}N(d_2)}{\sigma\sqrt{t}K} + 0.
$$

4. Proof of Corollary 1

We use the result summarized in Table 1.

(a) $\frac{\partial C}{\partial S} = e^{-qt}N(d_1) > 0$, hence C is increasing in S.

$\frac{\partial^2 C}{\partial S^2} = \frac{e^{-qt}N'(d_1)}{S\sigma\sqrt{t}} > 0$ implies C is convex in S.

(b) $\frac{\partial C}{\partial K} = -e^{-rt}N(d_2) < 0$, hence C is decreasing in K.

$\frac{\partial^2 C}{\partial^2 K} = \frac{e^{-rt}N'(d_2)}{\sigma\sqrt{t}K} > 0$, hence C is convex in K.

(c) $\frac{\partial C}{\partial r} = te^{-r}\ KN(d_2) > 0$, hence C is increasing in r.

$\frac{\partial C}{\partial q} = -te^{-qt}SN(d_1) < 0$, hence C is decreasing in q.

(d) $\frac{\partial C}{\partial \sigma} = S\sqrt{t}\,N'(d_1)e^{-qt} > 0$, hence hence C is increasing in σ.

For (e), $-\frac{\partial C}{\partial t} = -\frac{SN'(d_1)\sigma e^{-qt}}{2\sqrt{t}} + qSN(d_1)e^{-q} - re^{-rt}KN(d_2)$, which is a combination of negative and positive terms, and there is not enough information when $\frac{SN'(d_1)\sigma e^{-qt}}{2\sqrt{t}} + re^{-rt}KN(d_2)$ is greater than $qSN(d_1)e^{-q}$, hence the effect of the time to maturity on the price of the call option is ambiguous.

5. Proof of Corollary 2

We use the result summarized in Table 2.

(a) $\frac{\partial P}{\partial S} = e^{-q}\{N(d_1) - 1\} < 0$, hence P is decreasing in S,

(recall that $N(d_1)$ is always less than 1.)

$\frac{\partial^2 P}{\partial S^2} = \frac{e^{-qt}N'(d_1)}{S\sigma\sqrt{t}} > 0$ implies P is convex in S.

(b) $\frac{\partial P}{\partial K} = -e^{-r}\{N(d_2) - 1\} > 0$, hence P is increasing in K.

$\frac{\partial^2 P}{\partial^2 K} = \frac{e^{-rt}N'(d_2)}{\sigma\sqrt{t}K} > 0$, hence P is convex in K.

(c) $\frac{\partial P}{\partial r} = -te^{-rt}KN(-d_2) < 0$, hence P is decreasing in r.

$\frac{\partial P}{\partial q} = te^{-q}\,tSN(-d_1) > 0$, hence C is increasing in q.

(d) $\frac{\partial P}{\partial \sigma} = S\sqrt{t}\,N'(d_1)e^{-qt} > 0$, hence C is increasing in σ.

For (e), $-\frac{\partial P}{\partial t} = -\frac{SN'(d_1)\sigma e^{-qt}}{2\sqrt{t}} - qSN(-d_1)e^{-qt} + re^{-rt}KN(-d_2)$, which is a combination of negative and positive terms, and as is the case of call option, the effect of the time to maturity on the price of the put option is ambiguous.